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A NOTE TO THE UNIPOTENCY OF THE IDENTITY COMPONENT OF THE GROUP OF ALGEBRA AUTOMORPHISMS*

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Abstract

The properties of the group of automorphisms of local algebra are investigated in the following way: an algebra is called dwindlable, if there is an infinite sequence of automorphisms converging to the canonical epimorphism onto the underlying field; we confront this property with the possessing of a non-trivial torus of the identity component of the group of algebra automorphisms.

Introduction

We study local commutative finite dimensional algebras A over an infinite field K with characteristic 0, we also assume that they are split, i.e. $A/J_A = K$, where J_A is the Jacobson radical of A. Dominantly, the identification

$$K[x_1, \dots, x_n]/\mathfrak{i}$$

is used, $K[x_1, \dots, x_n]$ being the algebra of polynomials in n indeterminates over K and $\mathfrak{i}$ an ideal. (We write shortly *algebra* hereafter.) Special cases of such algebras are *Weil algebras* ($K = \mathbb{R}$) playing an important role in modern differential geometry. Geometric problems, namely classifications of natural lifts to bundles of contact elements, (cf. [3], [4]) also initiate the problem of a description of algebras A having the *fixed point subalgebra*

$$SA = \{a \in A; \phi(a) = a \text{ for all } \phi \in \operatorname{Aut} A\}$$

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trivial, i.e. isomorphic to K. Some known results about this problem and the correspondence of SA with natural lifts of geometric objects to bundles of contact elements are described in [3], [4] and [5].

We reckon two papers about automorphism groups of commutative algebras as key for our research: the first written by R. David Pollack [6] and the second written by Francisco Guil-Asensio and Manuel Saorín [1]. Both are indicative of the identity component G_A as a crucial tool for a systematic approach to properties of A.

For an algebra A in question, we call the *order* of A the minimum $\operatorname{ord}(A)$ of the integers r satisfying $J(A)^{r+1} = 0$. Further, the integer

$$\mathrm{w}(A) = \dim(J(A)/J^2(A))$$

is called the *width* of A. Let $\operatorname{Aut} A$ be the group of automorphisms of A, id_A the identical automorphism and $G_A \ni \mathrm{id}_A$ the connected *identity component* of $\operatorname{Aut} A$. A group which is simultaneously an algebraic set, i.e. a locus of zeros of a collection of polynomials, is called an *algebraic group*; $\operatorname{Aut} A$ is naturally an algebraic group. We have a canonical morphism of algebraic groups

$$\epsilon_A \colon \operatorname{Aut} A \to \mathrm{GL}(J(A)/J^2(A))$$

and its kernel U_A is then a closed subgroup consisting of merely *unipotent elements* (i.e. such automorphisms ϕ for which $\mathrm{id}_A - \phi$ is a nilpotent endomorphism of A); of course, $\mathrm{GL}(J(A)/J^2(A))$ reads as $\mathrm{GL}(\mathrm{w}(A), K)$. The group U_A is connected and hence contained in G_A.

Let $D(n, K)$ be the subgroup of diagonal matrices of the linear group $GL(n, K)$. When T is an algebraic group isomorphic to $D(m, K)$ for some m, then T is said to be a *torus*. The dimension of the maximal tori of an algebraic group is called the *rank*; if G_A does not contain any torus (G_A is of rank 0), then $G_A = U_A$. We recall some further results.

Proposition 1 ([1]). *A. The morphism ϵ_A is surjective if and only if $A = K[x_1, \dots, x_n]/\langle x_1, \dots, x_n\rangle^{r+1}$ for some $r \in \mathbb{N}$.*

B. The image of the morphism ϵ_A contains $D(n, K)$ if and only if A is monomial.

C. If an algebra A with $\mathrm{w}(A) = 2$ have $G_A = U_A$, then $\operatorname{ord}(A) \geq 4$.

As K is contained in A-algebra, the canonical algebra homomorphism $\kappa_A \colon A \to K$ can be viewed as the endomorphism $\kappa_A \colon A \to A$. We say that A is *dwindlable*, if there is an infinite sequence $\{\phi_n\}_{n=1}^{\infty}$ of automorphisms $\phi_n \in \operatorname{Aut} A$ such that $\phi_n \to \kappa_A$ for $n \to \infty$. For the sake of completion, we recall also some our existing results about the fixed point subalgebra

$SA = \{a \in A; \phi(a) = a$ for all $\phi \in \operatorname{Aut} A\}$, namely with respect to its triviality (the isomorphism with K).

Proposition 2 ([4], [5]). *A. If A is a dwindlable algebra, then SA is trivial. Nevertheless, there are also non-dwindlable algebras with trivial SA.*

B. If $U_A = G_A$, then SA can be both nontrivial and trivial.

New results

Propositions 1 and 2 formated conjectures giving a rise of this paper. We have obtained the following.

Proposition 3. *If A is dwindlable, then $G_A \supsetneqq U_A$.*

Proof. We have an infinite sequence $\{\phi_n\}_{n=1}^{\infty}$ of automorphisms $\phi_n \in \operatorname{Aut} A$ such that $\phi_n \to \kappa_A$ for $n \to \infty$. It is not restricting to consider all elements of this sequence as different. In general, individual automorphisms ϕ_n lie in diverse connected components of the group $\operatorname{Aut} A$, which are the cosets modulo the identity component G_A, cf. [1]. However, in a certain connected component H there is a subsequence $\{\phi_{n_k}\}_{k=1}^{\infty}$, $n_1 < n_2 < \ldots$, with the same limit. The component H is a coset modulo G_A in $\operatorname{Aut} A$, hence there exists an automorphism $\iota \in \operatorname{Aut} A$ such that $H = \iota(G_A)$. Applying ι^{-1}, we have the sequence $\{\iota^{-1}(\phi_{n_k})\}_{k=1}^{\infty}$. All automorphisms ϕ of A are of the form

$$\begin{array}{rcl} 1 & \stackrel{\phi^0}{\mapsto} & 1 \\ x_1 & \stackrel{\phi^1}{\mapsto} & P_1(x_1, \ldots, x_n) \\ & \cdots & \\ x_n & \stackrel{\phi^n}{\mapsto} & P_n(x_1, \ldots, x_n), \end{array} \tag{1}$$

where P_i are polynomials without absolute terms. It must be

$$\lim_{k\to\infty} \phi^1_{n_k} = \cdots = \lim_{k\to\infty} \phi^n_{n_k} = 0;$$

nevertheless, the automorphism ι^{-1} is of the form (1), too: it follows the sequence $\{\iota^{-1}(\phi_{n_k})\}_{k=1}^{\infty}$ has the limit κ_A as well and, moreover, all its elements belong to G_A. However it is impossible that all these elements are in the kernel with respect to ϵ. (Surely, only automorphisms with $P_i(x_1, \ldots, x_n) = x_i + Q_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$, are in this kernel, Q_i being polynomials without absolute and linear terms; indeed, it is impossible to choose a sequence converging to κ_A formed only with automorphisms as above.) Thus, $G_A \supsetneqq U_A$. □

Our second result is concentrated on algebras of the width 2. Their important role in algebraic research is known and we refer to [1] for a number of various results.

Proposition 4. *If A is an algebra with* $\mathrm{w}(A) = 2$ *and* $\mathrm{rank}(\mathrm{Aut}\, A) > 0$*, then A is dwindlable.*

Proof. It is easy to verify that homogeneous (i.e. the ideal $\mathfrak{i}$ in the expression $A = K[x_1, \dots, x_n]/\mathfrak{i}$ is homogeneous) algebras are dwindlable, cf. [4]. So, we assume A is non homogeneous. The condition $\mathrm{rank}(\mathrm{Aut}\, A) > 0$ reads as the connected identity component G_A of $\mathrm{Aut}\, A$ contains a nontrivial torus. It means that

$$\begin{aligned} \phi\colon \quad 1 &\overset{\phi^0}{\mapsto} \quad 1 \\ x &\overset{\phi^1}{\mapsto} \quad \alpha x \\ y &\overset{\phi^2}{\mapsto} \quad \beta y \end{aligned}$$

(where at most one of the coefficients α, β equals 1) is an automorphism (in a suitable basis). Let $\alpha \neq 1$. It is no restriction to assume $\alpha < 1$ (because we can take the inverse automorphism in the contrary case).

First, we look into the case $\beta \geq 1$. As A is non–homogeneous, there is a non–homogeneous binomial P with the following properties:

(i) $P \in \mathfrak{i}$

(ii) if $P = M_1 + M_2$ is a decomposition of P into monomials M_1, M_2, then $M_1 \notin \mathfrak{i}$ and $M_2 \notin \mathfrak{i}$

To show the existence of a P as above, we argue as follows: a non-homogeneous polynomial belonging to $\mathfrak{i}$, whose monomials are not belonging to $\mathfrak{i}$, satisfying (i) and (ii) exists in every set of generators of $\mathfrak{i}$; of course, it is a binomial in a suitable basis. As $\mathfrak{i} = \mathfrak{j} + J(A)^{\mathrm{ord}(A)+1}$, ($J(A)$ being the Jacobson radical generated by x and y and $\mathfrak{j}$ an ideal generated by polynomials of the order at most $\mathrm{ord}(A)$), the degree of such a binomial is less or equal $\mathrm{ord}(A)$. The monomials M_i are of the form $M_i = k_i x^{a_i} y^{b_i}$, $i = 1, 2$. If we evaluate $\phi(P)$, we obtain

$$\phi(M_1) = k_1 \alpha^{a_1} \beta^{b_1} x^{a_1} y^{b_1}, \quad \phi(M_2) = k_2 \alpha^{a_2} \beta^{b_2} x^{a_2} y^{b_2}.$$

As ϕ is an automorphism,

$$\alpha^{a_1} \beta^{b_1} = \alpha^{a_2} \beta^{b_2}$$

holds. If $\beta = 1$, then

$$a_1 = a_2 \text{ and } b_1 \neq b_2. \tag{2}$$

If $\beta > 1$, then

$$a_1 < a_2, b_1 < b_2 \text{ or } a_1 > a_2, b_1 > b_2. \tag{3}$$

We see that we can reorder (regardless of both incoming cases (2) and (3)) the monomials as follows:

$$P = \hat{M}_1 + \hat{M}_2,$$

where $\hat{a}_1 \leq \hat{a}_2$, $\hat{b}_1 < \hat{b}_2$ (we write $\hat{M}_i = \hat{k}_i x^{\hat{a}_i} y^{\hat{b}_i}$, $i = 1, 2$). It means $\deg \hat{M}_2 = \hat{a}_2 + \hat{b}_2 = \deg \hat{M}_1 + N = \hat{a}_1 + \hat{b}_1 + N$, $N \in \mathbb{N}$. Now, if we multiply P with

$$\frac{\hat{k}_2}{\hat{k}_1} x^{\hat{a}_2 - \hat{a}_1} y^{\hat{b}_2 - \hat{b}_1},$$

we obtain

$$P' = \hat{M}_1' + \hat{M}_2' = \hat{k}_2 x^{a_2} y^{b_2} + \frac{\hat{k}_2^2}{\hat{k}_1} x^{2\hat{a}_2 - \hat{a}_1} y^{2\hat{b}_2 - \hat{b}_1}.$$

Now, $\deg P' = \deg \hat{M}_2' = 2\hat{a}_2 - \hat{a}_1 + 2\hat{b}_2 - \hat{b}_1 = 2 \deg \hat{M}_2 - \deg \hat{M}_1 = \deg \hat{M}_1 + 2N$. It reads that the polynomial P' has higher degree than P. If $\hat{M}_2' \in \mathfrak{i}$, then also $\hat{M}_1' = \hat{M}_2 \in \mathfrak{i}$, a contradiction. If not, we use the binomial P' in place of P and repeat the procedure. The process stops with the mentioned contradiction because of the form of $\mathfrak{i}$. Hence $\beta < 1$.

Finally, we construct the infinite sequence of automorphisms from powers of ϕ. Evidently, $\{\phi^n\}_{n=1}^{\infty}$ (with $\alpha < 1$, $\beta < 1$) is coming to κ_A. □

We have the following corollary.

Proposition 5. *If A is an algebra with* $\mathrm{w}(A) = 2$ *and* $\mathrm{rank}(\mathrm{Aut}\, A) > 0$*, then SA is trivial.*

Proof. This follows directly from Proposition 2A and Proposition 4. □

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