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PAIRS OF PELL EQUATIONS HAVING AT MOST ONE COMMON SOLUTION IN POSITIVE INTEGERS

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Abstract

We prove that, for positive integers m and b, the number of simultaneous solutions in positive integers to $x^2-(4m^2-1)y^2=1$, $y^2-bz^2=1$ is at most one.

1 Introduction

As it is well known, a system of generalized Pell equations

$$ax^2-bz^2=\delta_1, \quad cy^2-dz^2=\delta_2, \tag{1}$$

where a, b, c, d are positive integers and δ_1, δ_2 are integers such that $\gcd(ab,\delta_1)=\gcd(cd,\delta_2)=1$, has at most finitely many solutions in positive integers provided that $d\delta_1\neq b\delta_2$ (see [26] or [25]).

If this is the case, one may effectively study the solutions of the system (1) by considering it as an elliptic equation $ac(xy)^2=(bz^2+\delta_1)(dz^2+\delta_2)$. This approach is useful in order to decide the existence of non-trivial solutions (see, for instance, [21]). To solve specific instances of (1), N. Tzanakis recommends an algorithm based on an elliptic equation associated to a pair of Pell equations. His algorithm uses lower bounds for linear forms in elliptic logarithms. A very well written exposition of these ideas with convincing examples and ample bibliography may be found in [27].

The components of any solution to (1) appear in second order recurrent sequences. Comparison of common terms in two such sequences leads to linear forms in the logarithms of three algebraic numbers. A. Baker's theory [3]

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allows one to effectively bound the solutions. The most general result of this type, due to E.M. Matveev [18], gives lower bounds for linear combinations of $n \geq 2$ logarithms. For three logarithms, better numerical coefficients have been devised by C.D. Bennett, J. Blass, A.M.W. Glass, D.B. Meronk and R.P. Steiner [6] and very recently by M. Mignotte [20]. Such bounds are useful in the quest of all solutions for specific values of the coefficients, but far from being sufficient, since the search space is still huge, defying the capabilities of present-day computational number theory. Any successful attempt to completely solve instances of (1) makes use of techniques from computational Diophantine approximations. Variations of the ground-breaking idea introduced by H. Davenport [4] are due to W.S. Anglin (see Section 4.6 of [2] for the description of an algorithm for solving some special forms of (1)), A. Dujella and A. Pethő [14], to name but a few contributors.

The hypergeometric method has also been instrumental in the study of simultaneous Pell equations. D.W. Masser and J.H. Rickert [17], followed by M.A. Bennett [7], applied techniques reposing on simultaneous Padé approximations of binomial functions. Skillful but, in principle, elementary arguments are sometimes successful in certain cases of (1), cf. [9], [29], [30], [31], [32], for instance.

Another natural question regarding Pell equations is to find how many solutions do exist. The results of A. Thue [26] and C.L. Siegel [25] establishing that (1) has finitely many solutions if $d\delta_1 \neq b\delta_2$ have ineffective proofs and provide no estimations for this number or for the size of solutions. P. Yuan [33, Theorem 2.1] shows that, for a, b, $c \geq 1$ and $\delta \in \{\pm 1, \pm 2, \pm 4, \}$ such that neither ab nor bc is a perfect square and $\gcd(abc, \delta) = 1$, the system

$$ax^2 - bz^2 = \delta, \quad cy^2 - bz^2 = \delta \tag{2}$$

has a positive integer solution if and only if each equation in (2) is solvable and ac is a perfect square. Moreover, in these circumstances, (2) has infinitely many solutions in positive integers. Masser and Rickert (*loc.cit.*) have given a procedure to obtain values for δ_1, δ_2 such that the number of common positive solutions to the generalized Pell equations $x^2 - 2z^2 = \delta_1$, $y^2 - 3z^2 = \delta_2$ is bigger than any prescribed bound.

More stringent results are known for particular cases. For the special form

$$x^2 - az^2 = 1, \quad y^2 - bz^2 = 1, \tag{3}$$

Anglin [1] showed that (3) has at most one positive solution provided that $b \leq 200$. The first general bound on the number of solutions to systems of Pell equations of this type has been given by H.P. Schlickewei [24], who proved that no more than $4 \times 8^{2^{78}}$ integer solutions exist. Masser and Rickert [17] bounded the number of positive solutions of (3) by 16. M.A. Bennett [7] lowered this bound to 3. Such a result is almost best possible, since for

$$n(l,m)=\frac{\alpha^{2l}-\alpha^{-2l}}{4\sqrt{m^2-1}},$$

with l and m integers greater than 1 and $\alpha=m+\sqrt{m^2-1}$, it is readily seen that $(x_1,y_1,z_1)=(m,n(l,m),1)$ and

$$(x_2,y_2,z_2)=\left(\frac{\alpha^{2l}+\alpha^{-2l}}{2},2n(l,m)^2-1,2n(l,m)\right)$$

are two positive integral solutions to (3) with $a=m^2-1$ and $b=n(l,m)^2-1$. More generally, if one defines $c(l,m)$ by

$$4c(l,m)-1=\frac{(\sqrt{m}+\sqrt{m-1}\,)^l-(\sqrt{m}-\sqrt{m-1}\,)^l}{2\sqrt{m-1}},$$

then, for $l\equiv 3 \pmod 4$, one has $c(l,m)$ integer and $(1,1,1)$ and

$$\left(\frac{(\sqrt{m}+\sqrt{m-1}\,)^l+(\sqrt{m}-\sqrt{m-1}\,)^l}{2\sqrt{m}},4c(l,m)-3,4c(l,m)-1\right)$$

are two positive solutions to $mx^2-(m-1)z^2=c(l,m)y^2-(c(l,m)-1)z^2=1$.

Since there are known no examples of simultaneous Pell equations with three solutions, it is widely believed that the tight bound for the number of solutions to (1) with $\delta_1=\delta_2=1$ is two. The most precise statement formalizing this generally accepted opinion appears in [33].

Conjecture. *There is at most one positive solution to the simultaneous Pell equations*

$$ax^2-by^2=1,\quad cy^2-dz^2=1, \tag{4}$$

with the exception of

$$(a,b,c,d)=\big(1,m^2-1,1,n(l,m)^2-1\big)$$

or

$$(a,b,c,d)=\big(m/a_0^2,(m-1)/b_0^2,c(l,m)/c_0^2,(c(l,m)-1)/d_0^2\big),$$

where a_0, b_0, c_0, d_0 are positive integers, when there exist two positive solutions.

Yuan succeeded to confirm this conjecture for systems of the form (3) with $\max\{a,b\}>14\cdot 10^{56}$. His result has been improved by M.A. Bennett, M.

Cipu, M. Mignotte and R. Okazaki [8] by proving that the above Conjecture is unconditionally true when $a = c = 1$ and $b \neq d$. (For a detailed proof, the reader may consult [12].) Similar results have been obtained for other particular cases of system (1). Quite recently, Cipu and Mignotte [13] have proved that the equations (4), where a, b, c and d are positive integers with $c \neq d$, $a > 1$, $b > 1$, have at most two positive solutions. In [13] it is also shown that the system of Diophantine equations

$$x^2 - ay^2 = 1, \quad y^2 - bz^2 = 1 \tag{5}$$

has at most two common solutions with x, y, $z > 0$.

The proofs of these results combine upper bounds for the putative solutions (obtained à la Baker) with suitable gap principles (assuring that consecutive solutions are rather far apart from each other).

For specific values of the coefficients, systems of the form (1) have been completely solved by A. Baker and H. Davenport [4], E. Brown [11], C.M. Grinstead [15], R.G.E. Pinch [22], among others. For further references, the reader is referred to [27].

The problem of estimating the number of solutions to the system of Diophantine equations (5) has been attacked by Yuan [32] using a different approach, based on properties of Lucas sequences. A key rôle play primitive prime divisors of Lucas numbers. A result of Ljunggren [16], according to which the Diophantine equation $Ax^2 - By^4 = 1$ has at most one positive solution if A, $B > 0$, allows Yuan to conclude that the system of equations (5) has at most one solution in positive integers for $a = 4m(m + 1)$.

The aim of this paper is to study systems of the type (5) not covered by Yuan's result.

Teorema 1.1. *If n and b are positive integers, then the simultaneous Pell equations $x^2 - (4m^2 - 1)y^2 = y^2 - bz^2 = 1$ have at most one solution (x, y, z) in positive integers.*

In the proof of this result linear forms in logarithms have no visible presence. In fact they are deep inside the proof of classification of Lehmer numbers with no primitive divisors. The structure of the paper is as follows. In Section 2 we study Lehmer sequences naturally attached to positive solutions of (5). Then we combine the ingredients to obtain a proof for Theorem 1.1.

2 From a solution to Lucas sequences

We aim to find common solutions for the pair of Pell equations

$$x^2 - (4m^2 - 1)y^2 = 1, \tag{6}$$

$$y^2 - bz^2 = 1, \tag{7}$$

where m, b are positive integers and b is not a perfect square. Taken separately, each of these equations has solutions expressible in terms of their respective fundamental solutions α and β and conjugates thereof $\overline{\alpha}$ and $\overline{\beta}$. Notice that we have

$$\alpha := 2m + \sqrt{4m^2 - 1},\ \overline{\alpha} := 2m - \sqrt{4m^2 - 1}. \tag{8}$$

Consider the Lucas sequences

$$U_k := \frac{\alpha^k - \overline{\alpha}^k}{\alpha - \overline{\alpha}},\ V_k := \frac{\alpha^k + \overline{\alpha}^k}{2},\ U'_k := \frac{\beta^k - \overline{\beta}^k}{\beta - \overline{\beta}},\ V'_k := \frac{\beta^k + \overline{\beta}^k}{2}. \tag{9}$$

Then any solution (x, y, z) of the system (6)–(7) has the form

$$x = V_k,\ y = U_k = V'_l,\ z = U'_l, \quad \text{for some positive integers } k \text{ and } l.$$

It will be necessary to partially decompose in factors the terms of the Lucas sequence $(U_k)_k$, so we also introduce a Lehmer sequence. Notice that $\alpha = \rho^2$ and $\overline{\alpha} = \overline{\rho}^2$, where

$$\rho := \frac{\sqrt{4m+2} + \sqrt{4m-2}}{2},\ \overline{\rho} := \frac{\sqrt{4m+2} - \sqrt{4m-2}}{2}. \tag{10}$$

Denote

$$x_k = \frac{\rho^k + \overline{\rho}^k}{\rho + \overline{\rho}},\ y_k = \frac{\rho^k - \overline{\rho}^k}{\rho - \overline{\rho}}, \quad \text{for } k \text{ odd}, \tag{11}$$

$$x_k = \frac{\rho^k + \overline{\rho}^k}{2},\ y_k = \frac{\rho^k - \overline{\rho}^k}{\rho^2 - \overline{\rho}^2}, \quad \text{for } k \text{ even}. \tag{12}$$

It is easy to see that we have $x_0 = 1$, $x_1 = 1$, $x_2 = 2m$, $x_3 = 4m - 1$, $y_0 = 0$, $y_1 = 1$, $y_2 = 1$, $y_3 = 4m + 1$, and the other terms are obtained from the linear recurrence relations

$$x_{k+2} = 4mx_k - x_{k-2},\ y_{k+2} = 4my_k - y_{k-2}, \quad \text{for } k \geq 2. \tag{13}$$

A simple computation yields

$$(2m+1)x_k^2 - (2m-1)y_k^2 = 2, \quad \text{for any odd } k, \tag{14}$$

$$x_k^2 - (4m^2 - 1)y_k^2 = 1, \quad \text{for any even } k. \tag{15}$$

Moreover,

$$U_k = \begin{cases} x_k y_k, & \text{for } k \text{ odd}, \\ 2x_k y_k, & \text{for } k \text{ even}. \end{cases} \tag{16}$$

The properties of Lucas sequences are rather well known. For ease of reference, in the next lemma are recalled divisibility properties we shall use subsequently. The proofs may be found in several places, for instance, in [23] and [19].

Lemma 2.1. *a) If* $d = \gcd(k, n)$*, then* $\gcd(U_k, U_n) = U_d$*.*

b) If $d = \gcd(k, n)$*, then* $\gcd(V_k, V_n) = V_d$*, if* k/d *and* n/d *are odd, and 1 otherwise.*

c) If $d = \gcd(k, n)$*, then* $\gcd(U_k, V_n) = V_d$*, if* k/d *is even, and 1 otherwise.*

d) If $U_k \neq 1$*, then* $U_k \mid U_n$ *if and only if* $k \mid n$*.*

e) If $k \geq 1$*, then* $V_k \mid V_n$ *if and only if* n/k *is an odd integer.*

f) If $q \geq 1$ *and* $0 \leq k \leq n$*, then*

$$U_{2qn+k} - U_k = 2V_{qn+k}U_{qn}, \quad U_{2qn-k} + U_k = 2V_{qn-k}U_{qn},$$

$$V_{2qn\pm k} - V_k = 2(m^2 - 1)U_{qn\pm k}U_{qn}, \quad V_{2qn\pm k} + V_k = 2V_{qn\pm k}V_{qn}.$$

In the proof of our main result, an essential rôle play primitive divisors of U_k. Y. Bilu, G. Hanrot and P. Voutier [10] succeeded to completely solve the problem of existence of primitive divisors for Lucas and Lehmer numbers. As a consequence of their results, we have the information given in the upcoming lemma. The first assertion follows by looking up Table 1 and Table 3 of [10]. The second part holds for any Lucas or Lehmer sequence.

Lemma 2.2. *Any term* U_k *of index* $k > 1$ *has a primitive divisor. If* p *is a primitive divisor of some* U_k*, then* p *divides another term* U_n *if and only if* k *divides* n*.*

We also need specific information for the Lehmer sequences introduced above.

Lemma 2.3. *a) If* k *is odd, the four integers* x_k*,* y_k*,* x_{k+2}*,* y_{k+2} *are pairwise coprime.*

b) If k *is even, then* $\gcd(x_k, x_{k+2}) = \gcd(y_k, y_{k+2}) = 1$*,*

$$\gcd(x_{k+2}, y_k) = \begin{cases} 2m & \text{for } k \equiv 0 \pmod 4, \\ 1 & \text{for } k \equiv 2 \pmod 4, \end{cases}$$

$$\gcd(x_k, y_{k+2}) = \begin{cases} 1 & \text{for } k \equiv 0 \pmod 4, \\ 2m & \text{for } k \equiv 2 \pmod 4. \end{cases}$$

Proof. a) By induction, it follows from (13) that $\gcd(x_k, x_{k+2})$ and $\gcd(y_k, y_{k+2})$ divide $\gcd(x_1, x_3)$ and $\gcd(y_1, y_3)$, respectively. Using equation (16), one sees that $\gcd(x_k, y_{k+2})$ divides $\gcd(U_k, U_{k+2})$, which is $U_1 = 1$ by Lemma 2.1a).

Similarly one gets $\gcd(x_{k+2}, y_k) = 1$.
It remains to compute $d := \gcd(x_{2t+1}, y_{2t+1})$. Notice that $x_{2t+1}y_{2t+1}$ is odd for any t. Therefore d divides the greatest common divisor of $(x_{2t+1} + y_{2t+1})/2 = U_{t+1}$ and $(y_{2t+1} - x_{2t+1})/2 = U_t$. Again by Lemma 2.1*a*), we conclude that $d \mid U_1 = 1$.

b) Put $k = 2t$. Then we have

$$\gcd(x_{2t}, x_{2t+2}) = \gcd(V_t, V_{t+1}) = 1 \quad \text{by Lemma 2.1}b),$$

$$\gcd(y_{2t}, y_{2t+2}) = \gcd(U_t, U_{t+1}) = 1 \quad \text{by Lemma 2.1}a).$$

Similarly, $\gcd(x_{2t}, y_{2t+2}) = \gcd(V_t, U_{t+1})$, which is $V_1 = 2m$ if $t + 1$ is even, and 1 otherwise (see Lemma 2.1*c*)). Finally, $\gcd(x_{2t+2}, y_{2t}) = \gcd(V_{t+1}, U_t)$ is either $V_1 = 2m$ or 1, according to whether t is even or odd. □

Lemma 2.4. *For any even integer k, V_k is odd. For any odd k, V_k is divisible by $2m$ and $V_k/(2m)$ is odd.*

Proof. From the linear recurrence relation $V_{k+2} = 4mV_{k+1} - V_k$, it readily follows that the terms of even, respectively odd, index are generated by

$$V_{k+4} = (16m^2 - 2)V_{k+2} - V_k$$

from the initial terms $V_0 = 1$, $V_2 = 8m^2 - 1$, respectively $V_1 = 2m$, $V_3 = 2m(16m^2 - 3)$. Hence, V_k is even if and only if $k = 2t - 1$. Moreover, the integers $z_t := V_{2t-1}/(2m)$ satisfy a second-order linear recurrence relation

$$z_{t+2} = (16m^2 - 2)z_{t+1} - z_t,\ t \geq 1,\ z_1 = 1,\ z_2 = 16m^2 - 3.$$

□

We end this section by quoting an old result of Ljunggren [16].

Lemma 2.5. *The equation $Ax^4 - By^4 = C$, with A, $B > 0$ and $C \in \{\pm 1, \pm 2\}$, has at most one solution in positive integers.*

3 Proof of Theorem

Assume that the equations (5) have at least one common solution in positive integers. Each such solution has the form

$$x = V_k,\ y = U_k = V'_l,\ z = U'_l \tag{17}$$

for suitable natural numbers k and l. Let k_0 be the smallest positive integer k for which relations (17) hold. Then

$$y_0 = U_{k_0} = V'_{l_0},\ z_0 = U'_{l_0} \tag{18}$$

for the corresponding l_0. Notice that $k_0 > 1$, otherwise $y_0 = 1$, and therefore $z_0 = 0$.

We shall show that, for any solution (x, y, z) for the system (5) given by (17), one has $k = k_0$. Note that y_0 divides y (cf. [32, Lemma 2.4]), so that k_0 divides k and l_0 divides l. Moreover, k/k_0 and l/l_0 are odd. This and Lemma 2.1 imply $z_0 \mid z$. Hence the leftmost side in the sequence of equalities

$$\frac{z^2}{z_0^2} = \frac{U_k^2 - 1}{U_{k_0}^2 - 1} = \frac{U_{k+1}U_{k-1}}{U_{k_0+1}U_{k_0-1}} = \frac{x_{k+1}x_{k-1}y_{k+1}y_{k-1}}{x_{k_0+1}x_{k_0-1}y_{k_0+1}y_{k_0-1}} \tag{19}$$

is an integer.

If $k_0 > 2$, then U_{k_0+1} and U_{k_0-1} have primitive divisors p and q respectively. The last assertion of Lemma 2.2 implies

$$k_0 + 1 \text{ divides } k + 1 \text{ or } k - 1 \text{ and } k_0 - 1 \text{ divides } k + 1 \text{ or } k - 1. \tag{20}$$

Note that (20) holds even for $k_0 = 2$ because U_3 has a primitive divisor by Lemma 2.2 and $k_0 - 1 = 1$.

We shall discuss separately two cases.

The case k_0 even. Then $\gcd(U_{k_0+1}, U_{k_0-1}) = \gcd(U_{k+1}, U_{k-1}) = 1$. Therefore, if $k_0 + 1$ divides $k + 1$ and $k_0 - 1$ divides $k - 1$, then U_{k+1}/U_{k_0+1} and U_{k-1}/U_{k_0-1} are coprime integers, whose product is a square. Hence, each of

$$\frac{x_{k+1}y_{k+1}}{x_{k_0+1}y_{k_0+1}} \quad \text{and} \quad \frac{x_{k-1}y_{k-1}}{x_{k_0-1}y_{k_0-1}}$$

is the square of an integer. But $\gcd(x_t, y_t) = 1$ for odd t, see Lemma 2.3a), so that $x_{k+1} = A^2 x_{k_0+1}$ and $y_{k+1} = B^2 y_{k_0+1}$ for certain positive integers A, B. By relation (14) we have $(2m+1)x_{k_0+1}^2 A^4 - (2m-1)y_{k_0+1}^2 B^4 = 2$ and $(2m+1)x_{k_0+1}^2 - (2m-1)y_{k_0+1}^2 = 2$. These relations show that both couples $(1, 1)$, (A, B) verify equation $(2m+1)x_{k_0+1}^2 X^4 - (2m-1)y_{k_0+1}^2 Y^4 = 2$. Lemma 2.5 yields $A = B = 1$. Since the sequence (x_t) is strictly increasing, one concludes $k = k_0$.

If $k_0 + 1$ divides $k - 1$ and $k_0 - 1$ divides $k + 1$ then U_{k-1}/U_{k_0+1} and U_{k+1}/U_{k_0-1} are coprime integers, whence both

$$\frac{x_{k+1}y_{k+1}}{x_{k_0-1}y_{k_0-1}} \quad \text{and} \quad \frac{x_{k-1}y_{k-1}}{x_{k_0+1}y_{k_0+1}}$$

are squares of integers. Using Lemma 2.3a) we obtain $x_{k+1} = A^2 x_{k_0-1}$ and $y_{k+1} = B^2 y_{k_0-1}$, for certain positive integers A, B. This implies that $(2m+1)x_{k_0-1}^2 A^4 - (2m-1)y_{k_0-1}^2 B^4 = 2$ and, by Ljunggren's result, it follows that $k = k_0 - 2$, in contradiction with $k \geq k_0$.

If k_0^2-1 divides $k+1$ then from $\gcd(U_{k_0+1}, U_{k_0-1}) = 1$ and $U_{k_0\pm 1} \mid U_{k+1}$ it follows $U_{k+1} = AU_{k_0+1}U_{k_0-1}$. Using this in relation (19), one concludes that AU_{k-1} is a perfect square. As $\gcd(U_{k+1}, U_{k-1}) = 1$, one has U_{k-1} coprime with A, so that $A = S^2$, $U_{k-1} = T^2$ for some positive integers S, T. Applying Lemma 2.3*a*) one gets $x_{k-1} = C^2$, $y_{k-1} = D^2$. Therefore, $(2m+1)C^4 - (2m-1)D^4 = 2$. From $(2m+1) - (2m-1) = 2$ and the result of Ljunggren cited above, it follows $x_{k-1} = 1$. The only even value of k for which this happens is $k = 2$, when also $k_0 = 2$.

A similar conclusion is reached when $k_0^2 - 1$ divides $k - 1$.

The case k_0 odd. The proof goes through four subcases, with slight differences due to fact that U_{k_0+1} and U_{k_0-1} are no more coprime.

For the beginning, we consider what happens when $k_0 + 1$ divides $k + 1$ and $k_0 - 1$ divides $k - 1$. Then $x_{k_0+1}y_{k_0+1} \mid U_{k+1}$ and $U_{k_0-1} \mid U_{k-1}$. If $k_0 \equiv 1 \pmod 4$, then $k \equiv 1 \pmod 4$ as well, and from (12) and Lemma 2.3*a*), resp. *e*), it follows that y_{k_0+1} divides y_{k+1}, resp. x_{k_0+1} divides x_{k+1}. Since $\gcd(U_{k+1}/U_2, U_{k-1}/U_2) = 1$, one necessarily has $x_{k+1} = A^2x_{k_0+1}$ and $y_{k+1} = B^2y_{k_0+1}$ for some positive integers A, B. Then relation (15) yields $x_{k_0+1}^2 - (4m^2-1)y_{k_0+1}^2 = 1$ and $x_{k_0+1}^2A^4 - (4m^2-1)y_{k_0+1}^2B^4 = 1$. By Lemma 2.5, this is only possible for $A = B = 1$, that is $x_{k+1} = x_{k_0+1}$. Therefore k equals k_0, if k_0 is congruent to 1 modulo 4 and $(k_0+1) \mid (k+1)$, $(k_0-1) \mid (k-1)$. When $k_0 \equiv 3 \pmod 4$, a similar reasoning invoking $k_0 - 1$ leads to the conclusion $k = k_0$.

Next look at the subcase $(k_0+1) \mid (k-1)$, $(k_0-1) \mid (k+1)$. If $k_0 \equiv 1 \pmod 4$, then $k \equiv 3 \pmod 4$, so that x_{k-1} is multiple of x_{k_0+1}. From x_{k_0-1} coprime with $y_{k-1}x_{k+1}y_{k+1}$ and $\gcd(y_{k-1}/y_{k_0+1}, U_{k+1}/U_{k_0-1}) = 1$, one gets $x_{k-1} = A^2x_{k_0+1}$ and $y_{k-1} = B^2y_{k_0+1}$ for some positive integers A, B. By equation (15), $x_{k_0+1}^2 - (4m^2-1)y_{k_0+1}^2 = 1$ and $x_{k_0+1}^2A^4 - (4m^2-1)y_{k_0+1}^2B^4 = 1$. This and Lemma 2.5 imply $x_{k-1} = x_{k_0+1}$. Hence, $k - 1 = k_0 + 1$. As (k_0-1) divides $(k+1)$, one obtains $k_0 = 5$ and $k = 7$. Then, according to relation (19), $U_8/U_4 = 2V_4$ is a square, which is impossible because V_4 is odd by Lemma 2.4. When $k_0 \equiv 3 \pmod 4$, we obtain similarly $k_0 = k + 2$, which contradicts $k \geq k_0$.

Now we assume $(k_0+1) \mid (k+1)$ and $(k_0-1) \mid (k+1)$. Then, by Lemma 2.1, $U_{k_0+1} \mid U_{k+1}$, $U_{k_0-1} \mid U_{k+1}$, and $\gcd(U_{k_0+1}, U_{k_0-1}) = U_2 = 4m$. Therefore $k \equiv 3 \pmod 4$ and $\gcd(U_{k+1}, U_{k-1}) = U_2$, as well. From relation (19), one infers $U_{k-1} = 2x_{k-1}y_{k-1} = 4mS^2$ for a certain integer S. Since $\gcd(x_{k-1}, y_{k-1}) = 1$ (cf. Lemma 2.1*c*)) and $2m$ divides x_{k-1} (see Lemma 2.4), one has $x_{k-1} = 2mA^2$ and $y_{k-1} = B^2$ for suitable positive integers A, B. Then relation (15) yields $4m^2A^4 - (4m^2-1)B^4 = 1$. Since $4m^2 - (4m^2-1) = 1$, Lemma 2.5 implies $A = B = 1$, whence $x_{k-1} = 2m$, $y_{k-1} = 1$, and therefore $k = 3$. This in turn implies $k_0 = 3$, as desired.

Finally, suppose that $k-1$ is divisible by both k_0+1 and k_0-1. Then $k \equiv 1 \pmod 4$, U_{k-1} is multiple of U_{k_0+1} and U_{k_0-1}. From Lemma 2.1, $\gcd(U_{k_0+1}, U_{k_0-1}) = \gcd(U_{k+1}, U_{k-1}) = U_2 = 4m$. This and relation (19) imply $U_{k+1} = 2x_{k+1}y_{k+1} = 4mS^2$ for a certain integer S. Since x_{k+1} and y_{k+1} are coprime by Lemma 2.1 and x_{k+1} is divisible by $2m$, one has $x_{k+1} = 2mA^2$, $y_{k+1} = B^2$ for certain positive integers A, B. As before, one concludes $A = 1$, whence $k-1 = 2$, which is not divisible by 4.

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