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FINITE SIMPLICIAL MULTICOMPLEXES

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Abstract

Simplicial multicomplexes are a very natural generalization of simplicial complexes. Indeed, instead to see a simplicial complex as a subset $\Delta \subset \mathcal{P}([n])$ we can think Δ as a subset of vectors in $\{0,1\}^n$, which satisfy the property: $(*)$ For any $F \in \Delta$ and any $G \in \{0,1\}^n$ such that $G \leq F$ it follows that $G \in \Delta$. Nothing can stop us to consider subsets $\Gamma \subset \mathbb{N}^n$ which have the property $(*)$. Such a set is called a simplicial multicomplex.

In this paper we shall focus on the case of finite multicomplexes. More precisely, we shall exploit the relation between a monomial ideal (which will correspond to a finite multicomplex) and its polarized ideal (which will correspond to a simplicial complex). Using this connexion, we can extend many constructions and definitions in the category of simplicial complexes to the category of finite simplicial multicomplexes, as: homology, shellability, duality theories etc.

In the first section we introduce the main definitions and constructions of multicomplexes. In the second section, we present what we understand by a homology theory of multicomplexes. In the third section we extend the notion of shellability for simplicial multicomplexes and I prove a criterion of shellability (similar to the case of simplicial complexes) which allows us to see the duality with the case of ideals with linear quotients. This observation give us the idea to introduce the notion of co-shellable (multi)complexes. In the fifth section we define the base ring and the Erhart ring of a multicomplex. In the last section we give some dual constructions in the category of multicomplexes and some results which extend the case of simplicial complexes.

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1 Finite simplicial multicomplexes

First of all, let us fix some notations:

- k is an arbitrary field and $S = k[x_1, \ldots, x_n]$ is the ring of polynomials over k. For any monomial ideal $I \subset S$, we denote by $G(I)$ the set of minimal generators of I.
- A vector $u \in \mathbb{N}^n$ will be written as $u = (u(1), \ldots, u(n))$. The module of u is the number $|u| := u(1) + \cdots + u(n)$.
- If $u, v \in \mathbb{N}^n$, we say that $u \leq v$ if $u(i) \leq v(i)$ for all $i = 1, \ldots, n$. Obviously, "$\leq$" is a partial order on $\mathbb{N}^n$.
- We denote by $e_i = (0, \ldots, 1, 0, \ldots, 0)$ the vectors of the canonical base of $\mathbb{N}^n$.
- If $u \in \mathbb{N}^n$, x^u is the monomial $x_1^{u(1)} x_2^{u(2)} \cdots x_n^{u(n)} \in S$.

Definition 1.1. *A finite subset* $\Gamma \subset \mathbb{N}^n$ *is called a* finite simplicial multicomplex *if for all* $a \in \Gamma$ *and all* $b \in \mathbb{N}^n$ *with* $b \leq a$*, it follows that* $b \in \Gamma$*. The elements of* Γ *are called* faces.

An element $m \in \Gamma$ *is called a* maximal facet *if it does not exist* $a \in \Gamma$ *with* $a > m$*; in other words, if* m *is maximal with respect to "*$\leq$*". We denote* $\mathcal{M}(\Gamma)$ *the set of maximal facets of* Γ.

If $a \in \Gamma$ *is a face, the dimension of* a *is the number* $\dim(a) = |a| - 1$*. The* dimension *of* Γ *is the number* $\dim(\Gamma) = \max\{\dim(u) | u \in \Gamma\}$*. A multicomplex* Γ *is called* pure *if all the maximal facets have the same dimension, equal to* $\dim(\Gamma)$.

Remark 1.2. *An arbitrary intersection and a finite union of finite multicomplexes are again multicomplexes. Therefore, the set of all finite multicomplexes in* $\mathbb{N}^n$ *is the family of closed sets in a topology on* $\mathbb{N}^n$*, called the finite-simplicial topology. The continuous functions in this topology are called finite-simplicial morphisms of multicomplexes. This aspect will not be studied in this paper.*

Remark 1.3. *Any finite multicomplex is determined by its maximal facets set,* $\mathcal{M}(\Gamma) = \{u_1, \ldots, u_r\}$*. In fact,*

$$\Gamma = \{b \in \mathbb{N}^n |\ b \leq u_i,\ for\ some\ i \in \{1, \ldots, r\}\ \}.$$

We write $\Gamma = \langle u_1, \ldots, u_r \rangle$ *and we say that* Γ *is the* multicomplex spanned by the vectors $u_1, \ldots, u_r$*. Obviously,* Γ *is the smallest multicomplex which contains* $u_1, \ldots, u_r$*.*

Definition 1.4. *Let k be an arbitrary field. If $\Gamma \subset \mathbb{N}^n$ is a finite multicomplex, the ideal of* non-faces *of Γ is the monomial ideal, denoted by I_Γ, in $k[x_1, \ldots, x_n]$, spanned, as k-vector space, by all monomial x^a with $a \in \mathbb{N}^n \setminus \Gamma$. In particular, the monomials x^a with $a \in \Gamma$ forms a k-basis of S/I_Γ.*

Obviously, I_Γ is an Artinian ideal (i.e. S/I_Γ is an Artinian ring). Conversely, if I is an Artinian ideal, then $\Gamma_I = \{a \in \mathbb{N}^n | \ x^a \notin \Gamma\}$ is a finite multicomplex and moreover $I_{\Gamma_I} = I$.

Remark 1.5. *The ideal of non-faces of a simplicial multicomplex and the Stanley-Reisner ideal of a simplicial complex are different. More precisely, if Δ is a simplicial complex and I its the Stanley-Reisner ideal of Δ and if J is the ideal of non-faces of Δ regarded as a finite multicomplex, then I is the ideal generated by the square-free minimal generators of J.*

For example, if $\Delta = \langle \{1, 2\}, \{2, 3\}\rangle$, then the Stanley-Reisner ideal is $I = \langle x_1x_3 \rangle$ and the non-faces ideal of Δ (as a multicomplex) is $J = \left\langle x_1^2, x_2^2, x_3^3, x_1x_3 \right\rangle$.

Proposition 1.6. *1. Γ has only one maximal facet a, if and only if I_Γ is an irreducible Artinian monomial ideal.*

2. Let $(\Gamma_j)_j$ be a finite family of multicomplexes. Then:

$$I_{\bigcap_j(\Gamma_j)} = \sum_j I_{\Gamma_j}, \quad I_{\bigcup_j(\Gamma_j)} = \bigcap_j I_{\Gamma_j}.$$

3. Let $\Gamma = \langle u_1, \ldots, u_r \rangle$ be a finite multicomplex. Then

$$I_\Gamma = P_{u_1} \cap P_{u_2} \cap \cdots \cap P_{u_r}$$

is the unique irredundant irreducible decomposition of I_Γ.

Proof. 1. Since $\Gamma = \langle a \rangle$, it follows that

$$I_\Gamma = (x^b | b \in \mathbb{N}^n, \ b(i) > a(i) \ for \ some \ i) = (x_i^{a(i)+1} | i = 1, \ldots n).$$

Conversely, if I is an irreducible monomial ideal, then I is generated by powers of variables (i.e $I = \left\langle x_i^{c(i)} | c(i) \geq 1 \right\rangle$) and, thus $\Gamma_I = \langle a \rangle$, where $a = c - (1, \ldots, 1)$.

2. It is as an easy exercise.

3. It is obvious from 1. and 2. □

Definition 1.7. *Let $\Gamma \subset \mathbb{N}^n$ be a finite multicomplex. The ideal of maximal facets of Γ, denoted by $I(\Gamma) \subset S = k[x_1, \ldots, x_n]$ is:*

$$I = I(\Gamma) = \langle x^a | a \ is \ a \ maximal \ facet \ in \ \Gamma \rangle .$$

Conversely, to an arbitrary monomial ideal $I \subset S$ we can associate the multicomplex

$$\Gamma = \Gamma(I) = \langle a | x^a \textit{ is a minimal generator of } I \rangle .$$

Also, if I is a monomial ideal, we can associate the polarized ideal I^0 which is a square-free monomial ideal. The simplicial complex of the facets of I^0 is called the polarized simplicial complex *of Γ and it is denoted by $\Delta^0(\Gamma)$. Obviously, I is Cohen-Macaulay (Gorenstein etc.) if and only if the same property holds for I^0.*

Remark 1.8. *If $\Gamma = \langle u_1, \ldots, u_r \rangle$ is a finite multicomplex and $m = \vee_{j=1}^r u_j$, then $\Delta^0(\Gamma)$ is a simplicial complex on a set of vertices labeled $\{v_1^1, \ldots, v_{m(1)}^1, \ldots, v_1^n, \ldots, v_{m(n)}^n\}$. There is a bijection between the faces of Γ and the faces of Δ which have the property: $v_i^j \in F \Rightarrow v_{i-1}^j \in F, \ldots, v_1^j \in F$. More precisely, if $u \in \Gamma$ is a face, the corresponding face in $\Delta^0(\Gamma)$ is*

$$F_u = \{v_1^1, \ldots, v_{u(1)}^1, \ldots, v_1^n, \ldots, v_{u(n)}^n\}.$$

If we make any change on Δ^0 (for example, if we take the complementary complex of Δ^0 or the Alexander dual complex etc.) using the above correspondence and renumbering the vertices, we can write down a new multicomplex which it will be called the complementary multicomplex *of Γ (the Alexander dual of Γ etc.). This idea will be explained later, in the section* 3.

Definition 1.9. *We say that the multicomplex $\Gamma' \subset \mathbb{N}^m$ is a* subcomplex *of $\Gamma \subset \mathbb{N}^n$ if there exists a canonical inclusion of $\mathbb{N}^m$ in $\mathbb{N}^n$ such that $\Gamma' \subset \Gamma \cap \mathbb{N}^m$. In particular, if $n = m$, we demand that $\Gamma' \subset \Gamma$. Obviously, any subcomplex of Γ' in $\mathbb{N}^m$ for $m < n$ corresponds to a subcomplex of Γ in $\mathbb{N}^n$ but many of such subcomplexes could exist.*

For example, if $\Gamma = \langle (1, 2, 2), (2, 1, 2) \rangle$ and $\Gamma' = \langle (1, 1), (0, 2) \rangle$ then Γ' is a subcomplex of Γ via the inclusions $(a, b) \mapsto (a, 0, b)$ and $(a, b) \mapsto (b, a, 0)$ of $\mathbb{N}^2$ in $\mathbb{N}^3$ (There are still more 3 *posibilities. Find them!)*

Definition 1.10. *Let $\Gamma \subset \mathbb{N}^n$ be a simplicial multicomplex and $a \in \Gamma$. The* link *of a in Γ is the set*

$$lk_\Gamma(a) = \{b \in \Gamma | a + b \in \Gamma\}.$$

Obviously, $lk_\Gamma(a)$ is also a simplicial multicomplex and a subcomplex of Γ.

The star *of Γ is the set*

$$star_\Gamma(a) = \{b \in \Gamma | a \vee b \in \Gamma\},$$

which is also a subcomplex of Γ. Obviously, $lk_\Gamma(a) \subset star_\Gamma(a)$.

Let $\Gamma \subset \mathbb{N}^n$ and $\Gamma' \subset \mathbb{N}^n$ be two finite multicomplexes. The join *of Γ with Γ', denoted by $\Gamma * \Gamma'$, is the multicomplex:*

$$\Gamma * \Gamma' = \{u + v|\ u \in \Gamma,\ v \in \Gamma'\}.$$

Note that it is not necessary for Γ and Γ' to be in the same $\mathbb{N}^n$. In the general case, if $\Gamma \subset \mathbb{N}^n$ and $\Gamma \subset \mathbb{N}^m$, it is enough to choose two canonical inclusions $\mathbb{N}^n \subset \mathbb{N}^N$ and $\mathbb{N}^m \subset \mathbb{N}^N$ and to consider Γ and Γ' as multicomplexes in $\mathbb{N}^N$. Obviously, in that case, $\Gamma * \Gamma'$ depends on the chosen inclusions. However, there is a canonical way to compute $\Gamma * \Gamma'$: It is enough to take $N = n+m$ and $N^n \subset N^{n+m}$ to be $(a(1), \ldots, a(n)) \mapsto (a(1), \ldots, a(n), 0, \ldots, 0)$, respectively $N^m \subset N^{n+m}$ to be $(b(1), \ldots, b(m)) \mapsto (0, \ldots, 0, b(1), \ldots, b(m))$.

In particular, if $\Gamma \subset \mathbb{N}^n$ is a multicomplex and $\Gamma' = \{0, 1\} \subset \mathbb{N}$, then $\Gamma * \Gamma'$ in the sense of the last construction, is called the cone *over Γ.*

Example 1.11. *Let $\Gamma = \langle (3, 1, 2), (2, 1, 3), (3, 2, 1) \rangle$. Then*

$$lk_\Gamma(2, 0, 0) = \langle (1, 1, 2), (0, 1, 3), (1, 2, 1) \rangle.$$

Also, $lk_\Gamma(3, 0, 0) = \langle (0, 1, 2), (0, 2, 1) \rangle$ and $star_\Gamma(3, 0, 0) = \langle (3, 1, 2), (3, 2, 1) \rangle$.

Proposition 1.12. *Let Γ be a finite multicomplex, $u \in \Gamma$ and $v \in lk_\Gamma(u)$. Then:*

1. $\dim(\Gamma) = \dim(lk_\Gamma(u)) + |u|$. *If Γ is pure, then $lk_\Gamma(u)$ is also pure.*
2. $u \in lk_\Gamma(v)$ *and* $lk_{lk_\Gamma(u)}(v) = lk_{lk_\Gamma(v)}(u) = lk_\Gamma(u + v)$.
3. $\langle v \rangle * lk_{lk_v(\Gamma)}(u) \subset lk_{star_\Gamma(v)}(u)$.
4. *If $\Gamma = \langle u_1, \ldots, u_r \rangle$, $u \in \Gamma$, and $a \in \mathbb{N}^n$, then:*

$$star_\Gamma(u) = \langle u_i | u \leq u_i \rangle,\ \ lk_\Gamma(u) = \langle u_i - u | u \leq u_i \rangle,$$

$$\langle a \rangle * \Gamma = \langle u_1 + a, \ldots, u_r + a \rangle.$$

Proof. 1.This is obvious.

2. $v \in lk_\Gamma(u)$ implies $v + u \in \Gamma$, that is $u \in lk_\Gamma(v)$. Let $w \in lk_{lk_\Gamma(u)}(v)$. Then $w + v \in lk_\Gamma(u)$, so $w + v + u \in \Gamma$, which is equivalent to the fact that $w \in lk_\Gamma(u + v)$.

We can rewrite this proof, easier, as follows: $lk_{lk_\Gamma(u)}(v) = \{w \in \mathbb{N}^n | v + w \in lk_\Gamma(u)\} = \{w \in \mathbb{N}^n | v + w + u \in \Gamma\} = lk_\Gamma(u + v)$. Analogously, $lk_{lk_\Gamma(v)}(u) = lk_\Gamma(u + v)$.

3. Let us suppose that $w \in \langle v \rangle * lk_{lk_\Gamma(v)}(u)$. Then $w = w' + w''$ with $w' \leq v$ and $\eta = w'' + u + v \in \Gamma$. We have to prove that $(w + u) \vee v \in \Gamma$.

Since $w' \leq w \wedge v \leq v$ and $w - w \wedge v \leq w''$, we can assume that $w' = w \wedge v$ and $w'' = w - w'$. Let $\eta := w - w \wedge v + u \in \Gamma$. It is enough to show that $(w+u) \vee v \in \Gamma$. We have

$$\eta(i) = \begin{cases} v(i) + u(i), & v(i) > w(i), \\ w(i) + u(i), & v(i) \leq w(i). \end{cases}$$

Let $\xi := (w+u) \vee v$. If $v(i) \leq w(i)$, then $v(i) \leq w(i) + u(i)$, thus $\xi(i) = w(i) + u(i)$. When $v(i) > w(i)$, we cannot say that $v(i) \geq w(i) + u(i)$ but, anyway, $\xi(i) \leq v(i) + u(i)$. The conclusion is that $\xi \leq \eta \in \Gamma$, therefore $\xi \in \Gamma$ as required.

4.The proof is an easy exercise. □

Example 1.13. *Let* $\Gamma = \langle (3,4,4), (4,2,5) \rangle$, $u = (3,2,1)$ *and* $v = (0,1,2)$. *Obviously,* $star_\Gamma(v) = \Gamma$. *Then* $lk_{star_\Gamma(v)}(u) = lk_\Gamma(u) = \langle (0,2,3), (1,0,4) \rangle$. *Since* $lk_{lk_\Gamma(v)}(u) = lk_\Gamma(u+v) = lk_\Gamma(3,3,3) = \langle (0,1,1) \rangle$, *we have* $\langle v \rangle * lk_{lk_\Gamma(v)}(u) = \langle (0,2,3) \rangle$. *This example shows that the inclusion* $\langle v \rangle * lk_{lk_v(\Gamma)}(u) \subset lk_{star_\Gamma(v)}(u)$ *can be strict (in the case of simplicial complexes, always, we have the equality).*

2 Geometrical description and homology of multicomplexes.

Definition 2.1. *Let* $\Gamma = \langle u_1, \ldots, u_r \rangle$ *be a finite simplicial multicomplex. Let* $\Delta^0 = \Delta^0(\Gamma)$ *be the polarized complex associated to* Γ. *Let* $|\Delta^0|$ *be the underlying topological space of* Δ^0. *As we already have seen,* Δ^0 *is a simplicial complex on a set of vertices labeled by* $\{v_1^1, \ldots, v_{m(1)}^1, \ldots, v_1^n, \ldots, v_{m(n)}^1\}$.

The topological space associated to Γ, *denoted by* $|\Gamma|$ *is the quotient topological space of* $|\Delta^0|$ *obtained by gluing the vertices* $\{v_1^1, \ldots, v_{m(1)}^1\}$, $\ldots$, *respectively* $\{v_1^n, \ldots, v_{m(n)}^1\}$.

Example 2.2. *If* $\Gamma = \langle a \rangle$ *with* $a \geq (1, \ldots, 1)$, *then* $|\Gamma| \sim \vee_{i=1}^s S^1$ *where* $s = |a| - n$. *This follows easily be induction on* $|a|$.

Example 2.3. *Let* $\Gamma = \langle (2,1), (1,2) \rangle \subset \mathbb{N}^2$. *The polarized simplicial complex of* Δ *is* $\Delta^0 = \left\langle \{v_1^1, v_2^1, v_1^2\}, \{v_1^1, v_1^2, v_2^2\} \right\rangle$. *(In other language,* $I = I_\Gamma = \left\langle x^2y, xy^2 \right\rangle$ *and the polarized ideal of* I *is* $I^0 = \langle x_1x_2y_1, x_1y_1y_2 \rangle$.*) For reasons of comprehensibility, we rewrite as* $\Delta^0 = \langle \{1,2,3\}, \{2,3,4\} \rangle$.

Note that $|\Delta^0|$ *consists in two triangles with the common edge* $\{2,3\}$. *Therefore,* $|\Gamma|$ *is the topological space obtained from* $|\Delta^0|$ *by gluing the vertices* 1 *with* 2 *and* 3 *with* 4 *respectively. The obtained topological space* $|\Gamma|$ *is homotopically equivalent with* $S^1 \vee S^1$.

In algebraic language, the gluing "corresponds" to the factorization with $x_1 - x_2$ *and* $y_1 - y_2$ *that gives the isomorphism:*

$$\frac{K[x_1,x_2,y_1,y_2]}{(x_1x_2y_1, x_1y_1y_2, x_1-x_2, y_1-y_2)} \cong \frac{k[x,y]}{(x^2y, xy^2)}.$$

Definition 2.4. *Let* $\Gamma \subset \mathbb{N}^n$ *be a finite simplicial multicomplex with* $e_i \in \Gamma$, $i = 1,\ldots,n$. *Let* A *be an arbitrary commutative ring with unity. Let* Δ^0 *be the polarized simplicial complexul associate to* Γ, *and let*

$$\{v_1^1,\ldots,v_{m(1)}^1,\ldots,v_1^n,\ldots,v_{m(n)}^n\}$$

be its vertices. Let $C_i(\Delta^0, A)$ *be the free* A*-module spanned by the set of* i*-faces of* Δ*. (This is the complex of* A*-modules which is used to compute the simplicial homology of* Δ^0*.)*

Let $C_i(\Gamma, A) := C_i(\Delta^0, A)$ *for* $i \geq 1$ *and let* $C_0(\Gamma, A) := C_0(\Delta^0, A)/(e_j^i - e_k^i)$, *where* e_j^i *is the base of* $C_0(\Gamma, A)$ *(more precisely,* e_j^i *corresponds to the vertex* v_j^i*). It is obvious that* $C_0(\Gamma, A) \cong A^n$.

Let $\partial_i : C_i(\Gamma, A) \to C_{i-1}(\Gamma, A)$, *for* $i \geq 2$ *be the usual differentials and let* $\partial_1 : C_1(\Gamma, A) \to C_0(\Gamma, A)$ *be the composed map*

$$C_1(\Gamma, A) = C_1(\Delta^0, A) \to C_0(\Delta^0, A) \to C_0(\Gamma, A).$$

Let $\partial_0 := 0$. *Obviously,* $\partial_{i-1} \circ \partial_i = 0$ *for all* $i \geq 1$.

The homology of $C_*(\Gamma, A)$ *is the* simplicial homology *of the simplicial multicomplex* Γ *and we denote it by* $H_*(\Gamma, A)$*. This means that*

$$H_i(\Gamma, A) = Ker(\partial_i)/Im(\partial_{i+1}).$$

Remark 2.5. *(Connection with algebraic topology) Let* Γ *be a finite simplicial multicomplex. The* i-skeleton *of* Γ *is the subcomplex* $\Gamma^{(i)} = \{a \in \Gamma |\ |a| \leq i\}$.

Let Γ *be a simplicial multicomplex. Then* $|\Gamma^{(i+1)}|$ *is obtained, topologically, by attaching some* $i+1$*-cells over* $|\Gamma^{(i)}|$*. Moreover, this gluing is compatible with the differentials* ∂_i*. In conclusion,* $|\Gamma|$ *has a structure of cellular complex which is identically with its simplicial structure. I.e. the complex* $C_*(\Gamma, A)$ *is exactly the cell complex of* A*-modules which computes the homology for a cellular complex. See Example* 2.7 *for further explanations.*

Corollary 2.6. *For any multicomplex* Γ, $H_*(\Gamma, A) = H_*(|\Gamma|, A)$.

Proof. One way to prove is simply using the above remark. Another way to prove this corollary is the following: Obviously, one has $H_*(\Delta, A) = H_*(|\Delta|, A)$ for any simplicial complex Δ. In particular this holds for $\Delta = \Delta^0(\Gamma)$. If X is a "nice" connected topological space (a topological variety for example) and

$x, y \in X$ and $x \sim y$ then $X/\sim$ is homotopically equivalent with $X \vee S^1$. Therefore, $|\Gamma| \approx |\Delta^0| \vee S^1 \vee \cdots \vee S^1$, where S^1 appears exactly $s-n$ times and $s = rang(C_0(\Delta^0))$. But,

$$H_*(|\Gamma|, A) \cong \begin{cases} H_i(\Delta^0, A), & for\ i \neq 1, \\ H_i(\Delta^0, A) \oplus A^{s-n}, & for\ i = 1. \end{cases}$$

Now it is obvious that $H_*(\Gamma, A) = H_*(|\Gamma|, A)$. □

Example 2.7. • *Let $\Gamma = \langle (3) \rangle \subset \mathbb{N}$. $\Delta^0(\Gamma)$ is the 2-simplex, thus $|\Delta^0|$ is a triangle. Therefore, $|\Gamma|$ is obtained from the triangle by gluing its vertices. ($|\Gamma|$ looks as a "parachute"!) Obviously, $|\Gamma| \sim S^1 \vee S^1$. Let us explain the structure of cell complex of $|\Gamma|$. The 0-skeleton consists in a point. The 1-skeleton consists in three circles glued in that point (that means that we have attached three 1-cells over the 0-skeleton). At last, we attached one 2-cell over those three circles to obtain $|\Gamma|$. Let us write down the simplicial homology (which is identically with the cell homology) of Γ:*

$$0 \longrightarrow A \xrightarrow{\partial_2} A^3 \xrightarrow{\partial_2} A \xrightarrow{\partial_0} 0.$$

Let us denote $C_2(\Gamma, A) = e_{123}A$, $C_1(\Gamma, A) = e_{12}A + e_{13}A + e_{23}A$, $C_0(\Delta^0, A) = e_1A + e_2A + e_3A$ and $C_0(\Gamma, A) = C_0(\Delta^0, A)/(e_1 - e_2, e_1 - e_3) = eA$, where e_{123} corresponds to the face $\{1, 2, 3\}$ of Δ^0 etc.

We have $\partial_2(e_{123}) = e_{23} - e_{13} + e_{12}$. Also, $\partial_1(e_{ij}) = \widehat{e_j} - \widehat{e_i} = e - e = 0$. Thus $\partial_1 = 0$. Since ∂_2 is injective, $H_2(\Gamma, A) = 0$. Also, $H_1(\Gamma, A) = Ker(\partial_1)/Im(\partial_2) = A^3/A = A^2$ and $H_0(\Gamma, A) = Ker(\partial_0)/Im(\partial_1) = A^3/A^2 = A$. This is the well known homology of $S^1 \vee S^1$!

- *Let $\Gamma = \langle (2, 1), (1, 2) \rangle$ be the multicomplex from the Example 1.17. We have already seen that $\Delta^0 = \langle \{1, 2, 3\}, \{2, 3, 4\} \rangle$ and that $|\Gamma| \sim S^1 \vee S^1$. Write down the homology of Γ. We have $C_2(\Gamma, A) = A^2$, $C_1(\Gamma, A) = A^5$, $C_0(\Gamma, A) = A^2$, so:*

$$0 \longrightarrow A^2 \xrightarrow{\partial_2} A^5 \xrightarrow{\partial_2} A^2 \xrightarrow{\partial_1} 0.$$

The matrix of ∂_2 is

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix}$$

and the matrix of ∂_1 *is*

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

Obviously, $rank(\partial_2) = 2$ *and* $rank(\partial_0) = 0$*. Then* $H^2(\Gamma, A) = 0$*, because* ∂_2 *is injective.* $H^1(\Gamma, A) = Ker(\partial_1)/Im(\partial_2) = A^4/A^2 = A^2$ *and, of course* $H^1(\Gamma, A) = A$*.*

The structure of cell complex for Γ *is the following: Since* $\Gamma^0 = \{(1,0),(0,1)\}$*, then* $|\Gamma^0|$ *consist in two points. Since* $\Gamma^1 = \Gamma_0 \cup \{(2,0),(1,1),(0,2)\}$*,* $|\Gamma^1|$ *is obtained from* $|\Gamma^0|$ *by attaching three 1-cells. The first cell (corresponding to* $(2,0)$*) is glued as a loop over the first point, the second cell (corresponding to* $(1,1)$*) is glued as a line between the points of* $|\Gamma^0|$ *and the third cell is glued as a bucle over the second point. Since* $\Gamma^2 = \Gamma_1 \cup \{(2,1),(1,2)\}$*,* $|\Gamma^1|$ *is obtained by gluind two discs, both of them with the border equal with* $|\Gamma_1|$*. The geometrical image of* $|\Gamma|$ *is a "parachute" (as in example above), and therefore* $|\Gamma^2|$ *is homotopically equivalent with* $S^1 \vee S^1$*.*

Remark 2.8. *(The reduced homology of a simplicial multicomplex) As in the case of the simplicial complexes, we can define the reduced homology for a multicomplex to be the homology of the following complex of A-modules:*

$$\cdots \to C_i(\Gamma, A) \to C_{i-1}(\Gamma, A) \to \cdots \to C_0(\Gamma, A) \to C_{-1}(\Gamma, A) = A \to 0,$$

where the last map ∂_0 *is given by the matrix* $(1, \dots, 1)$*. Obviously, we think* $C_{-1}(\Gamma, A)$ *as the free A-module generated by the* -1*-faces of* Γ*, (i.e. by (0,...,0)). We denote by* $\widetilde{H}_*(\Gamma, A)$ *the reduced homology of* Γ*. Of course,* $\widetilde{H}(\Gamma, A) = \widetilde{H}(|\Gamma|, A)$*.*

Remark 2.9. *For any multicomplex* Γ*, the cone over* Γ *is acyclic, i.e.* $\widetilde{H}_*(\Gamma, A) = 0$*. Indeed, as a topological space, the cone over* Γ *is obviously contractible, and therefore it has no reduced homology.*

Definition 2.10. *Let* $\Gamma \in \mathbb{N}^n$ *be a finite simplicial multicomplex and let A be an arbitrary commutative ring with unity. We consider the chain complex:*

$$\cdots \to C_i(\Gamma, A) \to C_{i-1}(\Gamma, A) \to \cdots \to C_0(\Gamma, A) \to 0.$$

Applying the functor $Hom(-, A)$ *to this complex, we obtained a cochain complex:*

$$0 \to Hom(C_0(\Gamma, A), A) \to Hom(C_0(\Gamma, A), A) \to \cdots$$
$$\ldots \to Hom(C_{i-1}(\Gamma, A), A) \to Hom(C_i(\Gamma, A), A) \to \cdots.$$

Let $C^i(\Gamma, A) := Hom(C_i(\Gamma, A), A)$*. We define the differentials*

$$\delta_i : C^i(\Gamma, A) \to C^{i+1}(\Gamma, A)$$

by

$$\delta_i(f)(x) := (-1)^i f(\partial_{i+1}(x)),$$

for any $x \in C^{i+1}(\Gamma, A)$*.*

The simplicial cohomology of Γ *is, by definition, the cohomology of the cochain complex above, i.e.* $H^i(\Gamma, A) := Ker(\delta_i)/Im(\delta_{i-1})$*. Moreover,* $H^*(\Gamma, A)$ *has a structure of a graded* A*-algebra with the cup-product.*

Of course, $H^*(\Gamma, A) = H^*(|\Gamma|, A)$ *and, as in the homological case, we can define, similarly, the reduced cohomology of* $|\Gamma|$*.*

Remark 2.11. *It would be interesting to compute the Euler characteristic* $\chi(|\Gamma|)$ *using only the combinatorial structure of* $\Gamma = \langle u_1, \ldots, u_r \rangle$*. Of course, it is obvious that* $\chi(|\Gamma|) = \chi(\Delta^0(\Gamma)) + n - |sup(\Gamma)|$*, where* $sup(\Gamma) = \vee_{i=1}^r u_i$*. So, the problem is to compute* $f_i(\Delta^0)$ *using the combinatorial structure of* Γ*.*

3 Shellable finite multicomplexes

Let us recall that a simplicial complex Δ is said to be connected if there exists an ordering on the facet set of Δ, $\{F_1, \ldots, F_r\}$, such that $F_i \cap F_{i+1} \neq \emptyset$. Obviously, Δ is connected if and only if $|\Delta|$ is a connected space. In the case of multicomplexes, we have the following generalization:

Definition 3.1. *A finite simplicial multicomplex* Γ *is said to be* connected*, if there exists an ordering on* $\mathcal{M}(\Gamma) = \{u_1, \ldots, u_r\}$ *such that* $u_i \wedge u_{i+1} > (0, \ldots, 0)$*, for* $i = 1, \ldots, r-1$*. Obviously,* Γ *is connected if and only if its underlying topological space is connected.*

We have the following well known characterisation of shellable simplicial complexes, see for example [2, Chapter 4].

Proposition 3.2. *Let* Δ *be a connected pure simplicial complex. Let* $F_1, \ldots, F_r$ *be a fixed ordering of the set of facets of* Δ*. Then, the following assertions are equivalent:*

1. Δ *is shellable whit the ordering* $F_1, \ldots, F_r$*: i.e* $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ *is generated by a set o proper maximal faces of* $\langle F_i \rangle$*.*

2. *The set* $S_i = \{F | F \in \langle F_1, \ldots, F_i \rangle, F \notin \langle F_1, \ldots, F_{i-1} \rangle\}$ *has only one minimal element, for any* $i = 2, \ldots, r$*.*

3. *For any* $j < i$*, there exists a vertex* $v \in F_i \setminus F_j$ *and there exists* $k < i$ *such that* $F_i \setminus F_k = \{v\}$*.*

Definition 3.3. *Let* $\Gamma \subset \mathbb{N}^n$ *be a finite multicomplex. Let* $b, a \in \Gamma$*. We call* a *a* lower neighbour *of* b *if there exists an integer* k *such that* $a(k) + 1 = b(k)$ *and* $a(i) = b(i)$ *for any* $i \neq k$*. Equivalently,* a *is a lower neighbour of* b *if* $a < b$ *and* $|a| = |b| - 1$*.*

For example, $(4, 3, 0, 2)$ *is a lower neighbor of* $(4, 3, 1, 2)$*.*

Definition 3.4. *Let* Γ *be a finite connected pure multicomplex. We say that* Γ *is* shellable *as a finite multicomplex, if there is an order on the set of maximal facets of* Γ*,* $u_1, \ldots, u_r$*, such that* $\langle u_1, \ldots, u_{i-1} \rangle \cap \langle u_i \rangle$ *is generated by a set of lower neighbours of* u_i*.*

Our first aim is to give a characterization of shellability for a multicomplex similar to the above proposition.

Proposition 3.5. *Let* Γ *be a finite connected pure multicomplex. The followings are equivalent:*

1. Γ *is shellable with the order* $u_1, \ldots, u_r$ *on* $\mathcal{M}(\Gamma)$*.*

2. *The set* $S_i = \{v \in \mathbb{N}^n | v \leq u_i, v \nleq u_j \ for \ j < i\}$ *has only one minimal element* v*, which, moreover, has the property* $v(j) = u_i(j)$ *or* $v(j) = 0$*,* $(\forall) j \in [n]$*.*

3. *For any* $j < i$*, there exists* $m \in [n]$ *with* $u_i(m) > u_j(m)$ *and* $k < i$ *such that* $u_i(m) = u_k(m) + 1$*, and* $u_i(s) \leq u_k(s)$ *for* $s \neq m, s \in [n]$*.*

Proof. $(1 \Rightarrow 2)$. Let us suppose that $\langle u_1, \ldots, u_{i-1} \rangle \cap \langle u_i \rangle$ is generated by the following lower neighbors of u_i, $u_i - e_{i_1}, \ldots, u_i - e_{i_k}$. Let

$$v := \begin{cases} u_i(j), & j \in \{i_1, \ldots, i_k\}, \\ 0, & j \notin \{i_1, \ldots, i_k\}. \end{cases}$$

It is enough to prove that v is a minimal element of $S_i = \{a \in \mathbb{N}^n | a \leq u_i, a \nleq u_j \ for \ j < i\}$. Obviously, $v \leq u_i$. Also, from its definition, $v \nleq u_j$ for $j < i$, because each u_j for $j \in \{i_1, \ldots, i_k\}$ has at least one of its components strictly less than $u_i(j)$.

Let us suppose now that there exists v' with $v' \leq u_i$ and $v' \nleq u_j$ for $j < i$. We have to show that $v \leq v'$.

Let us notice that the maximal facets of $\langle u_1, \ldots, u_{i-1} \rangle \cap \langle u_i \rangle$ are among $u_i \cap u_1, \ldots, u_i \cap u_{i-1}$. Also, since Γ is shellable, it follows that the maximal facets have the dimension $\dim(u_i) - 1$.

For any $j \notin \{i_1, \ldots, i_k\}$, we have $0 = v(j) \leq v'(j)$. Let us suppose that $v'(i_1) < v(i_1) = u_i(i_1)$. We choose j such that $u_j \wedge u_i = u_i - e_{i_1}$. We have $u_j(i_1) = u_i(i_1) - 1$ and $u_j(t) \geq u_i(t)$, for any $t \neq i_1$. But then $v' \leq u_j$ which is a contradiction.

$(2 \Rightarrow 3)$. Before giving the proof in the general case, let us study some particular cases. If $i = 1$ there is nothing to prove. If $i = 2$, we claim that there is only one nonzero component of f. Indeed, let suppose $v(1) = u_2(1) > 0, \ldots, v(e) = u_2(e) > 0$. Obviously, there is an index k such that $v(k) > u_1(k)$, since otherwise $v \leq u_1$, which is absurd. Let us suppose $v(1) > u_1(1)$. But then it is obvious that $v' = (v(1), 0, \ldots, 0) \in S_2$! This forces $e = 1$. From the uniqueness of v it follows that $u_1(k) \geq u_2(k)$, for any $k > 1$. Indeed, if $u_1(2) < u_2(2)$ for example, then $v' = (0, u_2(2), 0, \ldots, 0) \in S_2$ and this in a contradiction! We claim that $u_1(1) = u_2(1) - 1$. Indeed, if $u_1(1) \leq u_2(1) - 1$, then $v' = (u_2(1) - 1, 0, \ldots, 0) \in S_2$ and $v' < v$, which is again absurd. Since $|u_1| = |u_2|$, $u_1(1) = u_2(1) - 1$ and $u_1(k) \geq u_2(k)$ for any $k > 1$, it follows that there exists $m > 1$ such that $u_1(m) = u_2(m) - 1$ and $u_1(k) = u_2(k)$ for any $k \neq 1, m$. Thus, the assertion 3 holds.

Before to proceed to the general case, we make first some remarks:

- The condition 3 in the previous proposition can be replaced as follows: for any $j < i$ there exists $k < i$ such that $u_j \wedge u_i \leq u_k \wedge u_i$ and $d(u_i, u_k) = 1$.

- If $v \in S_i$ is the unique minimal element of S_i by reordering of the vertices, we can assume that
$$v(1) = u_1(1) > 0, \ldots, v(e) = u_i(e) > 0,\ v(e+1) = \cdots = v(n) = 0.$$

- For any $m > e$, there exists $j < i$ such that $u_i(m) \leq u_j(m)$. Indeed, otherwise, the vector $(0, \ldots, u_i(m), \ldots, 0)$ will be in S_i which is a contradiction whit the uniqueness of v.

- Also, we cannot have simultaneously $v(1) > \max\{u_1(1), \ldots, u_{i-1}(1)\}$ and $v(2) > \max\{u_1(2), \ldots, u_{i-1}(2)\}$ because, in this case, there are two minimal vectors in S_i.

- Last but not least, let us notice that the vectors u_j, for $j < i$, are obtained from a previous one be adding $+1$ to a component and subtracting $+1$ from another. A posteriori, this is clear from the definition of shellability. Anyway, this fact it is not used in the proof below.

Suppose $v = (u_i(1), \dots, u_i(e), 0, \dots, 0)$ is the unique minimal element of S_i. First of all, we want to prove that for any $j < i$, we have:

$$u_j \wedge u_i \leq (u_i(1) - 1, u_i(2), \dots, u_i(n)),$$

or

$$u_j \wedge u_i \leq (u_i(1), u_i(2) - 1, u_i(2), \dots, u_i(n))$$

or $\cdots$
or

$$u_j \wedge u_i \leq (u_i(1), \dots, u_i(e-1), u_i(e) - 1, u_i(e+1), \dots, u_i(n)).$$

But this is almost obvious! Indeed, if the above condition fails for some j, it follows that $v \leq u_j$.

Moreover, each inequality holds for some j. If, for example,

$$u_j \wedge u_i \not\leq (u_i(1) - 1, u_i(2), \dots, u_i(n)),$$

for any $j < i$, it follows that

$$(0, u_i(2), \dots, u_i(e), 0, \dots, 0) \in S_i,$$

which is a contradiction with the minimality of v.

Let $j < i$ with $u_j \wedge u_i \leq (u_i(1) - 1, u_i(2), \dots, u_i(n))$. We shall prove that there is $k < i$ such that $u_k \wedge u_i = (u_i(1) - 1, u_i(2), \dots, u_i(n))$ and this, obviously, completes the proof. Let us suppose that $u_j \wedge u_i \neq (u_i(1) - 1, u_i(2), \dots, u_i(n))$, for any $j < i$. Let $v' = (u_i(1) - 1, u_i(2), \dots, u_i(n))$. Obviously, $v' \leq u_i$. If there exists $k < i$ such that $v' \leq u_k$, it follows that $u_k \wedge u_i = (u_i(1) - 1, u_i(2), \dots, u_i(n))$, a contradiction. On the other hand, if $v' \not\leq u_j$, for any $j < i$, it follows that $v' \in S_i$, and this is again a contradiction, because $v \not\leq v'$!

($3 \Rightarrow 1$). Let $v \in \langle u_1, \dots, u_{i-1} \rangle \cap \langle u_i \rangle$. Then $v \leq u_i \wedge u_j$ for some $j < i$. Let m be as in assertion 3. Then, there exists k such that $u_i(m) = u_k(m) + 1$ and $u_i(s) \leq u_k(s)$ for $s \neq m$. Obviously, $v \leq u_k$, because $v \leq u_i$ (and then $v(s) \leq u_i(s) \leq u_k(s)$ for $s \neq m$) and $v(m) \leq u_j(m) \leq u_k(m)$. Thus $v \leq u_i \wedge u_k$. It is also clear that $|u_i \wedge u_k| = |u_i| - 1$. Then $u_i - e_m$ is a lower neighbor for u_i in $\langle u_1, \dots, u_{i-1} \rangle \cap \langle u_i \rangle$ cu $v \leq u_i - e_m$. But that means Γ is shellable. □

Example 3.6. *Let* $\Gamma = (2,1,0), (1,2,0), (0,2,1)$. *Then* Γ *is shellable. Indeed,* $\langle (1,2,0) \rangle \cap \langle (2,1,0) \rangle = (1,1,0)$ *and* $\langle (0,2,1) \rangle \cap \langle (2,1,0), (1,2,0) \rangle = (0,2,0)$.

The minimal element of $S_2 = \{v | v \leq u_1, v \not\leq u_2$ *is* $v = (0,2,0)$ *and the minimal element of* S_3 *is* $w = (0,0,1)$. *Obviously,* v *and* w *satisfy the condition* 2 *of the proposition.*

Remark 3.7. *Let Γ be a simplicial multicomplex and let $I(\Gamma)$ be the ideal of maximal facets of Γ. Suppose that Γ is shellable. From the assertion 3 of the proposition, we have: for any $j < i$ there exists $k < i$ such that $u_j \wedge u_i \leq u_k \wedge u_i$ şi $d(u_i, u_k) = 1$. The translation of this assertion in algebraic language is:*

For any $j < i$, there exists $k < i$ such that $gcd(m_i, m_j)|gcd(m_i, m_k)$ and $m_i/gcd(m_i, m_k) = x_t$ for some t. Note the similarity, but not coincidence, with the case of ideals with linear quotients!

Proposition 3.8. *Let Γ be a finite connected pure multicomplex. Then Γ is shellable if and only if $\Delta^0 = \Delta^0(\Gamma)$ is shellable.*

Proof. Suppose $\Gamma = \langle u_1, \dots, u_r \rangle$. Then $\Delta^0 = \langle F_1, \dots, F_r \rangle$, unde

$$F_i = \{v_1^1, \dots, v_{u_i(1)}^1, v_1^2, \dots, x_{u_i(2)}^2, \dots, x_1^n, \dots, x_{u_i(n)}^n\}.$$

Obviously Γ is pure, if and only if Δ^0 is pure. Assume that Γ is shellable. Using the above proposition, if follows that for any $j < i$, there exists m with $u_i(m) > u_j(m)$ and $k < i$ such that $u_i(m) = u_k(m) + 1$ and $u_i(s) \leq u_k(s)$ for $s \neq m$.

In terms of facets of Δ^0, the above fact is equivalent with the following one: For any $j < i$ there exists m with $x_{u_i(m)}^m \in F_i \setminus F_j$ and $k < i$ such that $F_i \setminus F_k = \{x_{u_i(m)}^m\}$. But this means that Δ^0 is shellable, as required. □

A well known property, see [2, Chapter 4], of shellable pure simplicial complexes is the following one:

Proposition 3.9. *If Δ is a pure shellable simplicial complex, then $|\Delta|$ has the homotopy type of a wedge of spheres of dimension d.*

Therefore, we have the following:

Corollary 3.10. *If Γ is a pure shellable multicomplex, then $|\Gamma|$ has the homotopy type of a topological space obtained by a wedge of spheres of dimension d by gluing some points and therefore, it is a wedge of spheres of dimension d and 1.*

We can extend the notion of shellability for the simplicial complexes which are not pure, as follows:

Definition 3.11. *Let Δ be a simplicial complex. Δ is called* shellable *if there exists an ordering of the facets of Δ, $F_1, \dots, F_r$, such that $\langle F_1, \dots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure of dimension* $\dim(\Delta) - 1$.

This definition can be extended for multicomplexes:

Definition 3.12. *Let* Γ *be a finite multicomplex.* Γ *is called* shellable *if there exists an ordering of the maximal facets of* Γ*,* $u_1, \dots, u_r$*, such that* $\langle u_1, \dots, u_{i-1}\rangle \cap \langle u_i\rangle$ *is generated by a set of lower neighbors of* u_i*.*

Lemma 3.13. *If* Δ *is shellable with the order* $F_1, \dots, F_r$*, then:* $|F_1| \geq |F_2|, \cdots, |F_1| \geq |F_r|$*. In particular,* $\dim(\Delta) = \dim(F_1)$*.*

Proof. We argue by induction on i. For $i = 2$, since $\langle F_1\rangle \cap \langle F_2\rangle$ has dimension $\dim(F_2)-1$, it follows that $|F_1| > |F_2|-1$ and, therefore, $|F_1| \geq |F_2|$. Suppose $i > 2$. Then, by the induction hypothesis, we have: $|F_1| \geq |F_2|, \cdots |F_1| \geq |F_{r-1}|$. Since $\langle F_1, \dots, F_{i-1}\rangle \cap \langle F_i\rangle$ has the dimension $\dim(F_i) - 1$, it follows that there exists $k < i$, such that $|F_k \cap F_i| = |F_i| - 1$. But then $|F_k| > |F_i| - 1$ which implies $|F_k| \geq |F_i|$, and $|F_1| \geq |F_i|$. □

This lemma can be written in language of multicomplexes:

Lemma 3.14. *If* Γ *is shellable with the order* $u_1, \dots, u_r$*, then:*

$$|u_1| \geq |u_2|, \cdots, |u_1| \geq |u_r|.$$

Proof. We argue by induction on i. For $i = 2$, since $\langle u_1\rangle \cap \langle u_2\rangle$ has dimension $\dim(u_2) - 1$, it follows that $|u_1| > |u_2| - 1$ and therefore $|u_1| \geq |u_2|$. Suppose $i > 2$. Then, by induction hypothesis, we have: $|u_1| \geq |u_2|, \cdots |u_1| \geq |u_{r-1}|$. Since $\langle u_1, \dots, u_{i-1}\rangle \cap \langle u_i\rangle$ has the dimension $\dim(u_i) - 1$, it follows that there exists a $k < i$, such that $|u_k \wedge u_i| = |u_i| - 1$. But then $|u_k| > |u_i| - 1$ which implies $|u_k| \geq |u_i|$, and $|u_1| \geq |u_i|$. □

Proposition 3.15. *Let* Δ *be a shellable simplicial complex. Then there exists a shelling order* $F_1, \dots, F_r$ *such that* $|F_1| \geq |F_2| \geq \cdots \geq |F_r|$*. Such a shelling is called a "good shelling".*

Proof. We use induction on r. For $r = 1, 2$ the assertion is obvious. Let us first prove the case $r = 3$. Suppose F_1, F_2, F_3 is a shelling with $|F_1| > |F_2|$ and $|F_3| > |F_2|$. From the definition of shellability, it follows that $F_3 \cap F_2 \subsetneq F_1 \cap F_3$ and $|F_1 \cap F_3| = |F_3| - 1$. We claim that F_1, F_3, F_2 is a good shelling.

Indeed, F_1, F_3 satisfy the definition of shellability. But we have $F_2 \cap F_3 \subset F_1 \cap F_3$. Taking the intersection with F_2, we get: $F_2 \cap F_3 \subset F_1 \cap F_3 \cap F_2$, and therefore $F_2 \cap F_3 \subset F_1 \cap F_2$.

In the general case, let us suppose that we have a shelling such that $|F_1| \geq |F_2| \geq \cdots \geq |F_{r-1}|$ and $|F_r| > |F_{r-1}|$. We choose the greatest j such that $|F_j| \geq |F_r|$ (i.e. $|F_{j+1}| < |F_r|$). We claim that $F_1, \dots, F_j, F_r, F_{j+1}, \dots, F_{r-1}$ is a good shelling Δ. Of course, the condition of shellability is satisfied from 1 to j. Let us show that $\langle F_1, \dots, F_j\rangle \cap \langle F_r\rangle$ is generated by $\dim(F_r)-1$-facets. But this is almost obvious: from hypothesis, we know that $\langle F_1, \dots, F_{r-1}\rangle \cap \langle F_r\rangle$

is generated by $\dim(F_r) - 1$-facets. Those facets are between $F_1 \cap F_r$, $\dots$, $F_{r-1} \cap F_r$. But $F_{j+1} \cap F_r, \dots, F_{r-1} \cap F_r$ have at most dimension $|F_r| - 2$.

Let us show that $\langle F_1, \dots, F_j, F_r\rangle \cap \langle F_{j+1}\rangle$ is generated by $\dim(F_{j+1}) - 1$-facets. It is sufficient to prove that $F_{j+1} \cap F_r$ is a subface of $F_{j+1} \cap F_t$ for some $t \leq j$. From the initial hypothesis ($F_1, \dots, F_r$ is a shelling), this is obvious, because $F_r \cap F_{j+1}$ cannot be a subface of $F_r \cap F_{j+s}$, $s > 1$ because it has to be included in a $\dim(F_r) - 1$-face.

Similarly, we prove the remained conditions. □

This lemma can be written in language of multicomplexes:

Proposition 3.16. *Let Γ be a shellable multicomplex. Then there exists a "good" shelling (i.e. a shelling with $|u_1| \geq |u_2| \geq \cdots \geq |u_r|$).*

Proof. As is the case of simplicial complexeles, we argue by induction on r, the cases $r = 1, 2$ being trivial. Let us suppose $r = 3$. We suppose $|u_1| > |u_2|$ and $|u_2| < |u_3|$. We claim that u_1, u_3, u_2 is a good shelling.

From definition of shellability, it follows that $F_3 \cap F_2 \subsetneq F_1 \cap F_3$ and $|F_1 \cap F_3| = |F_3| - 1$. We claim that F_1, F_3, F_2 is a good shelling.
Indeed, u_1, u_3 satisfy the definition of shellability. But we have $u_2 \wedge u_3 \leq u_1 \cap u_3$. Taking $\wedge u_2$, we get: $u_2 \wedge u_3 \subset u_1 \cap u_3 \cap u_2$, and therefore $u_2 \cap u_3 \subset u_1 \cap u_2$.

In the general case, let us suppose that we have a shelling such that $|u_1| \geq |u_2| \geq \cdots \geq |u_{r-1}|$ and $|u_r| > |u_{r-1}|$. We choose the greatest j such that $|u_j| \geq |u_r|$ (i.e. $|u_{j+1}| < |u_r|$). We claim that $u_1, \dots, u_j, u_r, u_{j+1}, \dots, u_{r-1}$ is a good shelling on Γ. Of course, the condition of shellability is satisfied from 1 to j. Let us show that $\langle u_1, \dots, u_j\rangle \cap \langle u_r\rangle$ is generated by $\dim(u_r) - 1$-maximal facets. But this is almost obvious: From hypothesis, we know that $\langle u_1, \dots, u_{r-1}\rangle \cap \langle u_r\rangle$ is generated by $\dim(u_r) - 1$-facets. Those facets are between $u_1 \cap u_r$, $\dots$, $u_{r-1} \cap u_r$. But $u_{j+1} \cap u_r, \dots, u_{r-1} \cap u_r$ have at most dimension $|u_r| - 2$.

Let us show that $\langle u_1, \dots, u_j, u_r\rangle \cap \langle u_{j+1}\rangle$ is generated by $\dim(u_{j+1}) - 1$-facets. It is sufficient to prove that $u_{j+1} \cap u_r$ is a subface of $u_{j+1} \cap u_t$ for some $t \leq j$. From the initial hypothesis ($u_1, \dots, u_r$ is a shelling), this is obvious, because $u_r \cap u_{j+1}$ cannot be a subface of $u_r \cap u_{j+s}$, $s > 1$ because it has to be included in a $\dim(u_r) - 1$-face.

Similarly, we prove the remained conditions. □

4 Co-shellable multicomplexes

In this section, all the complexes and multicomplexes are supposed pure.

Definition 4.1. *A simplicial complex* Δ *is called* co-shellable*, if there exists an order of the facets of* Δ*,* $F_1, \dots, F_r$*, such that:*

$$(*)\ \forall\ j < i, \exists\ v \in F_j \setminus F_i,\ si\ k < i\ cu\ F_k \setminus F_i = \{v\}.$$

Proposition 4.2. *Let* Δ *be a simplicial complex on the vertex set* $[n]$ *and let* $I = I(\Delta)$ *be the facet ideal of* Δ*. Then* I *has linear quotients if and only if* Δ *is co-shellable.*

Since the ideal of the basis of a matroid has linear quotients, it follows that any matroid is a pure co-shellable simplicial complex.

Proof. Let $I = (m_1, \dots, m_r)$ be a square-free monomial ideal. Let $\Delta = \langle F_1, \dots, F_r \rangle$ be the corresponding simplicial complex (i.e. $F_i = supp(m_i) \subset [n]$). We want to prove that Δ is co-shellable with that given ordering. Let $j < i$ and let $v = m_j / \gcd(m_i, m_j)$. Obviously, v is a square-free monomial. Since $v \cdot m_i = lcm(m_i, m_j)$, which is a multiple of m_j, it follows that $v \in (m_1, \dots, m_{i-1}) : m_i$. But I has linear quotients, and therefore there exists a variable $x_t | v$ such that $x_t \in (m_1, \dots, m_{i-1}) : m_i$. Then there exists a monomial m_k with $m_k | x_t m_i$. Thus $F_k \setminus F_i = \{t\}$, and $t \in F_j \setminus F_i$. This completes the proof. The converse has a similar proof. ☐

Example 4.3.

- *There exists shellable complexes which are not co-shellable. This is the case, for example, when we give a shelling* $F_1, \dots, F_r$ *such that* $F_i \cap F_j = \emptyset$ *for some* $j < i$*. For instance, let* Δ *be the complex of facets of the ideal* $I = (abc, bcd, def, efg)$*. Obviously,* Δ *is shellable, but* I *does not have linear quotients:* $(abc, bcd, def) : efg = (d, abc)$*.*
- *Even if we demand that* Δ *is* strong connected *(i.e. for any two facets* F_i *and* F_j *we have* $F_i \cap F_j \neq \emptyset$*) which is a very restrictive condition, there are shellable complexes which are not co-shellable. For example, if* Δ *is the facet complex of the ideal* $I = (abc, bcd, cde, cef)$*, then* Δ *is shellable but* I *does not have linear quotients:* $(abc, bcd, cde) : cef = (d, ab)$*.*
- *Also, there are co-shellable complexes which are not shellable. For instance, if* $\Delta = \langle abc, bcd, acd, ade, bce \rangle$*. It is easy to see that* $I(\Delta)$ *has linear quotients, but, also,* Δ *is not shellable since* $\langle bce \rangle \cap \langle abc, bcd, acd, ade \rangle$ *is not pure.*

The above definition can be extended for simplicial multicomplexes.

Definition 4.4. *A finite multicomplex* Γ *is called* co-shellable *if there exists an order of the maximal facets of* Γ *such that for any* $j < i$ *there is* m *and* $k < i$ *such that* $u_j(m) > u_i(m)$*,* $u_k(m) = u_i(m) + 1$*, and* $u_k(s) \leq u_i(s)$ *for* $s \neq m$*.*

Proposition 4.5. *Any monomial ideal I, generated by monomials of the same degree, has linear quotients if and only if the simplicial multicomplex of maximal facets of I is co-shellable.*

In particular, any discrete polymatroid is a co-shellable finite multicomplex.

Proof. The proof is the same as in the square-free case. Let $I = (m_1, \dots, m_r)$ be a monomial ideal and let $\Gamma = \langle u_1, \dots, u_r \rangle$ be the corresponding simplicial complex (i.e. $m_i = x^{u_i}$). We want to prove that Γ is co-shellable with that given order. Let $j < i$ and let $v = m_j / \gcd(m_i, m_j)$. Since $v \cdot m_i = lcm(m_i, m_j)$, which is a multiple of m_j, it follows that $v \in (m_1, \dots, m_{i-1}) : m_i$. But I has linear quotients, and therefore, there exists a variable $x_t | v$ such that $x_t \in (m_1, \dots, m_{i-1}) : m_i$. But that means that there exists a monomial m_k with $m_k | x_t m_i$. Thus $u_k(t) = u_i(t) + 1$ and $u_k(s) \leq u_i(s)$, for $s \neq t$. Also, since $x_t | v = m_j / \gcd(m_i, m_j)$ it results $u_j(t) > u_i(t)$. But this proves that Γ is co=shellable.

The converse implication has a similar proof. □

Proposition 4.6. *Let Δ be a simplicial complex. Then Δ is shellable if and only if Δ^c is co-shellable (where Δ^c is the complementary simplicial complex of Δ).*

Proof. Suppose Δ is shellable, i.e. there exists an order $F_1, \dots, F_r$ on the set of facets of Δ such that: for each $j < i$, there exists $v \in F_i \setminus F_j$ and there exists $k < i$ such that $F_i \setminus F_k = \{v\}$. We claim that $F_1^c, \dots, F_1^c$ is a co-shelling on Δ^c. But this is obvious, for the same choice of $k < i$ and v, since $F_j^c \setminus F_i^c = F_i \setminus F_j$ and $F_k^c \setminus F_i^c = F_i \setminus F_k$! □

Later, we will extend this property to multicomplexes.

5 The base ring and the Erhart ring of a multicomplex

Let Γ be a finite multicomplex with the set of maximal facets $\mathcal{M}_\Gamma = \{u_1, \dots, u_r\}$. The *base ring* of Γ is the monomial subalgebra

$$K[\mathcal{M}(\Gamma)] := k[x^{u_1}, \dots, x^{u_r}] \subset k[x_1, \dots, x_n].$$

The *Erhart ring* of Γ is the monomial subalgebra:

$$K[\Gamma] := k[x^u t | u \in \Gamma] \subset k[x_1, \dots, x_n, t].$$

Obviously, $K[\Gamma]$ is the semigroup ring of the cone over Γ,

$$C(\Gamma) = \langle (u_1, 1), \dots, (u_r, 1) \rangle .$$

We have a natural epimorphism $\varphi : B = k[t_1, \ldots, t_r] \to K[\mathcal{M}(\Gamma)]$, defined by $\varphi(t_i) := x^{u_i}$. If we take on B the grading, $\deg(t_i) := \deg(m_i)$, where $m_i = x^{u_i}$, then φ becomes a graded morphism. The kernel $Ker(\varphi) := P_{\mathcal{M}(\Gamma)}$ is called the toric ideal of $K[\mathcal{M}(\Gamma)]$. It is well known that $P_{\mathcal{M}(\Gamma)}$ is a graded prime ideal generated by a finite set of binomials. Of course, the same construction can be made for the Erhart ring.

It would be of great interest to find combinatorial conditions on Γ such that the base ring or the Erhart ring are normal, Cohen-Macaulay, Gorenstein etc. For example, if Γ is shellable, what can we say about $k[\mathcal{M}(\Gamma)]$ or $k[\Gamma]$?

6 Dual multicomplexes

Definition 6.1. *Let Δ be a simplicial complex. We called the* complementary complex *of Δ, and we denoted it by Δ^c, the complex*

$$\Delta^c = \langle [n] \setminus F | F \text{ } is \text{ } a \text{ } facet \text{ } of \text{ } \Delta \text{ } \rangle .$$

Obviously, if we think Δ as a subset of $\{0,1\}^n$, then

$$\Delta^c = \langle (1, 1, \ldots, 1) - F | F \in \Delta \text{ } facet \rangle .$$

We can, therefore, give the following generalization.

Definition 6.2. *Let $\Gamma \subset \mathbb{N}^n$ be a finite simplicial multicomplex with the set of maximal facets $\mathcal{M}(\Gamma) = \{u_1, \ldots, u_r\}$. If $u \in \mathbb{N}^n$ is an upper bound of Γ (i.e. $u \geq a$, for any $a \in \Gamma$; or equivalent: $\Gamma \subset \langle u \rangle$), then the complementary multicomplex of Γ with respect to u, denoted by Γ_u^c is the following one:*

$$\Gamma_u^c = \langle u - u_i | u_i \in \mathcal{M}(\Gamma) \rangle .$$

Γ_u^c depends on the choice of $u \in \mathbb{N}^n$. Of course, the least upper bound of Γ, which will be denoted by $sup(\Gamma)$, is $\sup(\Gamma) = \vee_{i=1}^r u_i$, where $\Gamma = \langle u_1, \ldots, u_r \rangle$. We denote $\Gamma_{\sup(\Gamma)}^c$ by Γ^c.

Remark 6.3. *Let Γ be a simplicial multicomplex and let $\Delta = \Delta^0(\Gamma)$ be the polarized simplicial complex of Γ. Let us consider Δ^c the complementary complex of Δ. Then, the multicomplex of ordered faces (see section 1) of Δ^c is Γ^c itself. The proof is obvious.*

Proposition 6.4. *If $\Gamma = \langle u_1, \ldots, u_r \rangle$ is a simplicial multicomplex, and $u \in \mathbb{N}^n$ is an upper bound of Γ, then u is an upper bound of Γ_u^c too, and:*

$$(\Gamma_u^c)_u^c = \Gamma.$$

Proof. If $\Gamma = \langle u_1 \rangle$ and $u \geq u_1$, the assertion is obvious, even in the case $u = u_1$. Let suppose $\Gamma = \langle u_1, \dots, u_r \rangle$ with $r \geq 2$. We claim that the only thing we have to prove is: if $a, b \in \mathbb{N}^n$ are two incomparable vectors, and $u \in \mathbb{N}^n$ $u > a$, $u > b$ then $u - a$,$u - b$ are incomparable. If the claim is true, then it follows that Γ_u^c has exactly r maximal facets $u - u_1, \dots u - u_r$, (and it is obvious that each of them is $\leq u$) and, therefore, $(\Gamma_u^c)_u^c$ has the maximal facets $u_1, \dots, u_r$. Thus $(\Gamma_u^c)_u^c = \Gamma$, as required.

The claim is almost clear. Indeed, if $u - a \geq u - b$, it follows $u(i) - a(i) \geq u(i) - b(i)$, for any $i = 1, \dots, n$, so $a(i) \leq b(i)$ for any i so $a \leq b$, which is a contradiction. □

In monomial language, we can write down the following definition:

Definition 6.5. *Let $I = (m_1, \dots, m_r)$ be a monomial ideal and let $\Gamma = \Gamma(I) = \langle u_1, \dots, u_r \rangle$ be the multicomplex of maximal facets of I. Let $u \in \mathbb{N}^n$ be an upper bound of Γ. (i.e. $lcm(m_1, \dots, m_r)|x^u$). The complementary ideal of I, with respect to x^u is the ideal $I_u^c := \langle x^u/m_i | i = 1, \dots, n \rangle$. I_u^c is the ideal of maximal facets of Γ_u^c.*

Example 6.6. *If $\Gamma = \langle (2,1,3), (1,2,3), (3,2,2) \rangle$ and $u = (4,4,3)$, then*

$$\Gamma_u^c = \langle (2,3,0), (3,2,0), (1,2,1) \rangle .$$

In algebraic language, if $I = (x^2yz, xy^2z^3, x^3y^2z^2)$ and $m = x^4y^4z^3$, then

$$I_m^c = (x^2y^3, x^3y^2, xy^2z).$$

Example 6.7. *If Γ is a multicomplex and $v \geq u \geq sup(\Gamma)$ are two vectors in $\mathbb{N}^n$, then one easily sees that $\Gamma_v^c = \langle v - u \rangle * \Gamma_u^c$.*

Proposition 6.8. *Let Γ be a pure multicomplex and $u \geq sup(\Gamma)$. Then Γ is shellable if and only if Γ_u^c is co-shellable.*

Proof. The case $\Gamma = \langle u \rangle$ is trivial. Since Γ is shellable, there exists an ordering of the maximal facets of Γ, $u_1, \dots, u_r$ such that: for any $j < i$, there exists m and $k < i$ such that: $u_i(m) > u_j(m)$, $u_i(m) = u_k(m) + 1$, and $u_i(s) \leq u_k(s)$, for $s \neq m$. We claim that Γ_u^c is co-shellable with the ordering of the maximal facets: $u - u_1, \dots, u - u_r$. Indeed, if we take m and $k < i$ as above, it is obvious that $(u - u_i)(m) = u(m) - u_i(m) < (u - u_j)(m) = u(m) - u_j(m)$ and $(u - u_i)(m) = (u - u_k)(m) - 1$ and $(u - u_i)(s) \geq (u - u_k)(s)$ for any $s \neq m$, as required. □

We will focuse now on the very important notion of Alexander duality. First of all, let us see what is the Alexander dual for a simplicial complex and how we can extend this concept in the case of multicomplexes.

Definition 6.9. *Let* Δ *be a simplicial complex. The* Alexander dual *of* Δ*, is the complex*

$$\Delta^{\vee} = \{[n] \setminus F | F \notin \Delta\}.$$

Thinking Δ *as a subset of* $\{0,1\}^n$*, we note that* $\Delta^{\vee} = \{(1,\dots,1) - F| \ F \in \{0,1\}^n \setminus \Delta\}$*. This gives us the idea of the following generalization:*

Definition 6.10. *Let* Γ *be a simplicial multicomplex and let* $u \in \mathbb{N}^n$ *be an upper bound of* Γ*. The* Alexander dual *of* Γ *w.r.t.* u *is the following multicomplex:*

$$\Gamma_u^{\vee} = \{u - v | v \leq u \ \ si \ \ v \notin \Gamma\}.$$

If $u = \sup(\Gamma)$*, we denote* $\Gamma_u^{\vee} =: \Gamma^{\vee}$*.*

Let us recall some results on the Alexander dual (in the case of simplicial complexes) which will be generalized in the case of multicomplexes. See [6] for details.

Proposition 6.11. *Let* Δ *be a simplicial complex on the set of vertices* $[n]$*. Let* I_{Δ} *be the Stanley-Reisner ideal of* Δ *and* $I(\Delta)$ *be the ideal of facets of* Δ*. Then:*

1. $(\Delta^{\vee})^{\vee} = \Delta$.
2. $I_{\Delta^{\vee}} = I(\Delta^c)$.
3. Δ *is shellable if and only if* $I_{\Delta^{\vee}}$ *has linear quotients.*

In the case of multicomplexes we have the following:

Proposition 6.12. *If* Γ *is a multicomplex and* $u \in \mathbb{N}^n$ *is an upper bound for* Γ*, then* $(\Gamma_u^{\vee})_u^{\vee} = \Gamma$*.*

Proof. Let us first note that we have an anti-monotone bijection between $\Gamma_u^{\vee}$ and the set $\{v \in \mathbb{N}^n | v \leq u \text{ and } v \notin \Gamma\}$. That means that we have a bijection between $(\Gamma_u^{\vee})_u^{\vee}$ and $\{v \in \mathbb{N}^n | v \leq u \text{ and } v \notin \Gamma_u^{\vee}\}$. But this last set is obvious in bijection with Γ. Thus $(\Gamma_u^{\vee})_u^{\vee} = \Gamma$, as required. □

Proposition 6.13. *If* Γ *is a multicomplex and* $u \in \mathbb{N}^n$ *is un upper bound for* Γ *then* $I_{\Gamma} = I(\Gamma_u^{\vee})_u^c$*, where* $u = \sup(\Gamma) + (1,\dots,1)$*. In particular,* $I_{\Gamma_u^{\vee}} = I(\Gamma_u^c)$*.*

Proof. Let us notice that $\Gamma_u^{\vee}$ is generated by $u - v$, where v is a minimal non-face of Γ. But the minimal non-faces of Γ are exactly the minimal generators of the ideal I_{Γ}. Writing this facts in algebraic language, we get:

$$I_{\Gamma} = \langle x^v | v \text{ is a minimal non-face of } \Gamma \rangle .$$

Also,

$$I(\Gamma_u^{\vee}) = \left\langle x^{u-v} | v \text{ is a minimal non-face of } \Gamma \right\rangle,$$

and, therefore, $I_\Gamma = I(\Gamma_u^{\vee})_u^c$, as required.

The last identity is clear when we replace Γ by $\Gamma_u^{\vee}$. □

Example 6.14. *If* $\Gamma = \langle (1,3),(4,2) \rangle$ *and* $u = (5,4) = sup(\Gamma) + (1,1)$*, then*

$$\Gamma_{(5,4)}^{\vee} = \langle (5,0),(0,4),(3,1) \rangle .$$

(This is easy to compute if we figure Γ *and* $\Gamma_{(5,4)}^{\vee}$ *on the same picture.) Also,*

$$(\Gamma_{(5,4)}^{\vee})_{(5,4)}^c = \langle (5,0),(0,4),(2,3) \rangle .$$

$I_\Gamma = (x^5, y^4, x^2y^3)$*. Obvious,* $I(\Gamma_{(5,4)}^{\vee})_{(5,4)}^c) = I_\Gamma$*.*

Corollary 6.15. *Let* Γ *be a multicomplex. Then* Γ *is shellable if and only if* $I_{\Gamma_u^{\vee}}$ *has linear quotients, where* $u = \sup(\Gamma) + (1,\dots,1)$*.*

Proof. If is obvious from Proposition 6.8 and Proposition 6.13. □

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