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COMPUTATIONS IN WEIGHTED POLYNOMIAL RINGS

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Abstract

In this note we survey some results which are useful to perform algebraic computations in a weighted polynomial ring.

Introduction and notation

In this survey paper we consider non-standard graded polynomial rings and take into examination some results concerning weighted Hilbert functions, weighted lexicographic ideals and Castelnuovo-Mumford regularity, from a computational point of view.

In Section 1 we recall some relevant facts about Hilbert functions of graded modules over a weighted polynomial ring. In particular we illustrate a method to compute the Hilbert function and the Hilbert polynomials given the associated Poincare series. In the second part we briefly discuss lexicographic ideals in the non-standard setting. In Section 2 we verify the validity of the operation of polarization in the non-standard case. In the final section we give a detailed proof of a formula which relates Castelnuovo-Mumford regularity with graded Betti numbers and establishes a natural counterpart to a very well-known and useful fact valid in the standard-case.

We consider polynomial rings over an infinite field K of characteristic 0 where

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the degrees of the variables are assumed to be positive integers with no further restriction. Variables are ordered by increasing degree (*weight*). We denote the polynomial ring by $R = K[\mathbf{X_1}, \dots, \mathbf{X_n}]$, where $\mathbf{X_i} = (X_{i1}, \dots, X_{il_i})$, $\deg X_{ij} = q_i$ for $j = 1, \dots, l_i$, and $q_1 < q_2 < \dots < q_n$. We let w be the weight vector $(\deg X_{11}, \dots, \deg X_{nl_n})$ so that (R, w) stands for a polynomial ring with the graduation given by w. We consider term orderings $>$ which are degree compatible and assume $X_{ij} > X_{ik}$ if $j < k$, $i = 1, \dots, n$. The total number of variables $\sum_{i=1}^{n} l_i$ will be denoted by l and the least common multiple of the weights $\mathrm{lcm}(q_1, \dots, q_n)$ by q. In Section 3 it is not necessary to group together variables of the same weight and therefore we re-label them $X_1, \dots, X_l$.

1 Hilbert functions

As in the standard graded case, homogeneous ideals can be studied by means of Hilbert functions. If M is a graded (R, w)-module, the assignment

$$H_M(s) \doteq \dim_K M_s$$

defines the *Hilbert function* $H_M : \mathbb{Z} \to \mathbb{N}$ of M, while the *Poincare series* of M is defined by

$$P(M, t) \doteq \sum_{i \geq 0} H_M(i) t^i \in \mathbb{Z}[\![t]\!].$$

It is well-known that the Poincare series of M can be expressed as a rational function

$$h(t) / \prod_{i=1}^{n} (1 - t^{q_i})^{l_i},$$

where $h(t) \in \mathbb{Z}[t]$. Recall that a function $G : \mathbb{Z} \to \mathbb{C}$ is called *quasi-polynomial (of period g)* if there exists a positive integer g and polynomials $p_0, \dots, p_{g-1}$ such that for all $s \in \mathbb{Z}$ one has $G(s) = p_j(s)$, where $s = hg + j$ and $0 \leq j \leq g - 1$. Thus, if I is a homogeneous ideal of (R, w), there exists a uniquely determined quasi-polynomial function $G_{R/I}$ such that $H_{R/I}(s) = G_{R/I}(s)$ for all $s \gg 0$. To be more precise, if we let d to be the order of the pole of $P(R/I, t)$ at $t = 1$, then there exist q polynomials $p_0, \dots, p_{q-1} \in \mathbb{Q}[t]$ of degree at most $d - 1$ and with coefficients in $[q^{d-1}(d-1)!]^{-1}\mathbb{Z}$ such that, for all $s \gg 0$,

$$H_{R/I}(s) = p_j(s) \quad \text{for } s \equiv j \bmod q.$$

Following the approach of [B], we now explain how to read the Hilbert polynomials from the Poincare series of R/I. If we let $P(R/I, t) = f(t)/g(t)$, with $f(t), g(t) \in \mathbb{Q}[t]$, from the division algorithm $f(t) = r(t)g(t) + s(t)$ we get a unique decomposition $P(R/I, t) = \mathrm{P_{pol}} + \mathrm{P_{rat}}$, where $\mathrm{P_{pol}} \in \mathbb{Q}[t]$ and

$\mathrm{P}_{\mathrm{rat}} \in \mathbb{Q}[\![t]\!]$, such that either $\deg \mathrm{P}_{\mathrm{rat}} < 0$ or $\mathrm{P}_{\mathrm{rat}} = 0$. Clearly, if $\mathrm{P}_{\mathrm{rat}} = 0$ then all of the Hilbert polynomials are zero. Moreover, one can show (cf. [B], Section 2) that there exist integers λ_{ij} such that

$$\mathrm{P}_{\mathrm{rat}} = \sum_{i=1}^{d} \sum_{j=0}^{q-1} \lambda_{ij} \frac{t^j}{(1-t^q)^i}.$$

Thus, if we let $\varphi_0(t) \doteq 1$ for all t, and $\varphi_i(t) \doteq (i!)^{-1}(t+1)\cdots(t+i)$, one can express the Hilbert polynomials of R/I by means of the following formula

$$p_j(s) = \sum_{i=1}^{d} \lambda_{ij} \varphi_{i-1} \left(\frac{s-j}{q} \right) \quad \text{for} \quad s \equiv j \bmod q.$$

The last few considerations allow us to compute the Hilbert function of R/I given the Poincare series as an input. On the other hand, this method is not optimal because it amounts to solve a linear system associated with a $dq \times dq$ matrix with integer entries. In order to improve the above reasoning, one can argue as follows. Let $\mathrm{P}_{\mathrm{rat}} = p(t)/q(t)$, where $p(t)$ and $q(t)$ are polynomials in $\mathbb{Z}[t]$ with no common factor, i.e. with no common complex roots. Let $\omega_1, \ldots, \omega_m$ be the distinct roots of $q(t)$ with multiplicities, we say, $d_1, \ldots, d_m$. Since $q(t)$ divides $g(t)$, it is clear that, for all i, ω_i is a root of the unity and $\omega_i^q = 1$. By using partial fractions, we know there exist unique ν_{ik}, with $i = 1, \ldots, m$ and $k = 1, \ldots, d_i$ such that

$$\mathrm{P}_{\mathrm{rat}} = \sum_{i=1}^{m} \sum_{k=1}^{d_i} \frac{\nu_{ik}}{(1-\omega_i t)^k}.$$

The coefficients ν_{ik} can be computed solving an in general much smaller linear system with coefficients in $\mathbb{Q}[\omega_1, \ldots, \omega_m]$. Since

$$\frac{1}{(1-\omega_i t)^k} = \sum_{s \geq 0} \binom{k+s-1}{k-1} (\omega_i t)^s,$$

and, therefore,

$$\begin{aligned} \mathrm{P}_{\mathrm{rat}} = \sum_{i=1}^{m} \sum_{k=1}^{d_i} \frac{\nu_{ik}}{(1-\omega_i t)^k} &= \sum_{i=1}^{m} \sum_{k=1}^{d_i} \sum_{s \geq 0} \nu_{ik} \binom{k+s-1}{k-1} (\omega_i t)^s \\ &= \sum_{s \geq 0} \left(\sum_{i=1}^{m} \sum_{k=1}^{d_i} \nu_{ik} \binom{k+s-1}{k-1} {\omega_i}^s \right) t^s \end{aligned}$$

we can use the fact that $\omega_i^s = \omega_i^{s \bmod q}$ to compute the Hilbert polynomials as follows

$$p_j(s) = \sum_{i=1}^{m} \sum_{k=1}^{d_i} \nu_{ik} \binom{k+s-1}{k-1} {\omega_i}^j$$

where $j = 0, \dots, q-1$ and $s \equiv j \bmod q$.
We implemented these formulas as functions for the computer algebra system [CoCoA] and the interested reader can download the code* at the URL
`http://www.dm.unipi.it/~dalzotto/HilbertNonStandard.coc`,
which contains also some procedures for computing weighted generic initial ideals and lexicographic ideals. We conclude this section by spending a few words about a way of testing whether a homogeneous ideal $I \subset (R, w)$ is *lexifiable*, i.e. admits an associated lexicographic ideal $I^{\text{lex}} \subset (R, w)$. It is easy to see that this is not always the case. What is needed is a method to check whether a monomial ideal is lexicographic, since in the non-standard case the ideal generated by a lexsegment needs not to be a lexicographic ideal.

In the following definition we denote $\sum_{i=1}^{l} w_i$ by $|w|$. For any non empty subset J of $\{1, \dots, l\}$, we let $|J|$ denote the cardinality of J and $q_J \doteq \operatorname{lcm}\{w_i\}_{i \in J}$.

Definition 1.1. Let $w \in \mathbb{N}^l_{>0}$. Then

$$G(w) \doteq \begin{cases} -w_1 & \text{if} \quad l = 1 \\ -|w| + \frac{1}{l-1} \sum_{2 \le \nu \le l} \left[\binom{l-2}{\nu-2}^{-1} \sum_{|J|=\nu} q_J \right] & \text{if} \quad l > 1. \end{cases}$$

It is not difficult to see that $G(w)$ can be computed recursively as follows

$$\sum_{i=1}^{l} G((w_1, \dots, w_{i-1}, \widehat{w_i}, w_{i+1}, \dots, w_l)) = (l-1)G(w) - q,$$

where $q = \operatorname{lcm}(w_1, \dots, w_l)$.
Knowing that, if (R, w) is a weighted polynomial ring and $n > G(w)$, each monomial of R_{n+hq} is divisible by a monomial in R_{hq} for any $h \in \mathbb{N}$ (cf. [BR] Proposition 4B.5), one can show the following result:

Proposition 1.2 ([DS], Proposition 4.9)**.** *Let $I \subset (R, w)$ be a homogeneous ideal generated in degree $\le d$ and let $q \doteq \operatorname{lcm}(q_1, \dots, q_n)$. If I_i is spanned (as a K-vector space) by a lexsegment for all $i \le d + q + G(w)$, then I is a lexicographic ideal.*

*the looks of which might appear quite sophisticated. This is due to the fact that [CoCoA] does not allow a straightforward use of algebraic extensions $\mathbb{Q}[\alpha]$ of $\mathbb{Q}$. A possible solution is to take normal forms with respect to the ideal generated by the minimal polynomial of α in $\mathbb{Q}[t]$.

2 Polarization

Polarization is a well-known algebraic operation on a monomial ideal which returns a squarefree monomial ideal in a larger polynomial ring. With some abuse of notation, we refer to polarization also when we consider a procedure which has been developed in [P] and consists of three fundamental operations on a homogeneous ideal I, which are polarizing a monomial ideal, modding out by a generic sequence of linear forms and taking initial ideals with respect to the lexicographic order. In the standard case, polarization returns as an output the associated lexicographic ideal of I. Here we show that one can define polarization also for homogeneous ideals in weighted polynomial rings and that the algorithm terminates when an ideal, which we denote by $I^{\mathbf{P}}$ and call *the complete polarization of* I, is computed. This needs not to be the lexicographic ideal associated with I, for instance because the last one does not necessarily exist.

Definition 2.1. Let $I \subseteq (R, w)$ be a monomial ideal and let P be the polynomial ring $K[Z_{ijh}]$ graded by $\deg Z_{ijh} \doteq q_i$, where $1 \le i \le n$, $1 \le j \le l_i$, $1 \le h \le N$, $N \gg 0$. Let $\pi : P \to R$ be the homogeneous map (of degree 0) defined by $\pi(Z_{ijh}) \doteq X_{ij}$. Then we call the monomial ideal of P generated by

$$\left\{ z^{p(\mu)} = \prod_{i=1}^{n} \prod_{j=1}^{l_i} \prod_{h=1}^{\mu_{ij}} Z_{ijh} \quad : \quad x^\mu = X_{11}^{\mu_{11}} \cdots X_{nl_n}^{\mu_{nl_n}} \text{ is a minimal generator of } I \right\}$$

the *polarization* of I and denote it by $I^{\mathbf{P}}$.

Observe that $I^{\mathbf{P}}$ is a squarefree ideal of P and that the graduation on P is chosen in such a way that the degree of $z^{p(\mu)}$ is the same as that of x^μ. Thus, I and $I^{\mathbf{P}}$ have minimal generators in the same degrees. In order to prove that all of the graded Betti numbers of I and $I^{\mathbf{P}}$ are the same, we recall the following result.

Lemma 2.2. *Let M be a finitely generated graded (R, w)-module. Let $f \in R_d$ be an M-regular form and $S \doteq R/(f)$. If $F_\bullet$ is a minimal graded free resolution of M, then $F_\bullet \otimes_R S$ is a minimal graded free resolution of M/fM as an S-module. In particular, the graded Betti numbers of M and M/fM are the same.*

Proof. Tensoring $F_\bullet \to M \to 0$ with S we obtain the complex of free S-modules $F_\bullet \otimes_R S \to M/fM \to 0$. We have to prove that all of the $\operatorname{Tor}_j^R(M, S)$ vanish. We achieve this by tensoring the resolution $0 \to R(-d) \to R \to S \to 0$ of S as an R-module with M, obtaining the complex $M(-d) \to M \to M/fM \to 0$. But this is exact, since f is M-regular and, consequently $\operatorname{Tor}_j^R(M, S) = \operatorname{Tor}_1^R(M, S) = 0$ for all j. □

Lemma 2.3. *The graded Betti numbers of I and $I^{\mathbf{P}}$ are the same.*

Proof. Let us fix i, j with $1 \leq i \leq n$ and $1 \leq j \leq l_i$. Let $S \doteq R[Z]$ with $\deg Z \doteq q_i$ and $\tau : S \to R$ be the graded ring homomorphism defined by $Z \mapsto X_{ij}$. Now consider the sets $A \doteq \{X^\mu \colon X^\mu \in G(I),\, X_{ij} \nmid X^\mu\}$ and $B \doteq \left\{\frac{X^\nu Z}{X_{ij}} \colon X^\nu \in G(I),\, X_{ij} \mid X^\nu\right\}$, where $G(I)$ denotes the minimal set of generators of I. Finally, let I' be the ideal of S generated by $A \cup B$.
It is easy to see that the polarization can be computed after a finite number of such steps. Therefore, by virtue of the previous Lemma, we only need to prove that $Z - X_{ij}$ is an S/I'-regular element. Suppose now that $Z - X_{ij}$ is not regular, i.e. $Z - X_{ij}$ belongs to an associated prime of I', we say $I' : m$, where $m \notin I'$. Since $I' : m$ is a monomial ideal, both $Z \in I' : m$ and $X_{ij} \in I' : m$. Therefore $Zm \in I'$ and $m \notin I'$. Thus, Zm is a multiple of some generator of I' of the form $\frac{Z}{X_{ij}} X^\mu$ and $Z \nmid m$. Since $X_{ij}m \in I'$ and $Z \nmid X_{ij}m$, $X_{ij}m$ is divisible by some $X^\mu \in A$. Finally, $X^\mu \mid m$ and $m \in I'$, which is a contradiction. $\square$

Let now $W = \{f_{ijh}\}$ be a collection of homogeneous polynomials of (R, w) with $\deg f_{ijh} = q_i$, $1 \leq i \leq n$, $1 \leq j \leq l_i$ and $1 \leq h \leq N$. Let $\sigma_W : P \to R$ be the homogeneous map (of degree 0) given by $\sigma_W(Z_{ijh}) = f_{ijh}$ and $I_W \doteq \sigma_W(I^{\mathbf{P}})$. If (R, w) is standard graded, then W is a collection of linear forms. It is known from [P] that, for a generic collection L of linear forms, I_L and I have the same graded Betti numbers. This fact can be easily generalized to the non-standard case, where instead of a generic collection of linear forms we use a generic collection of homogeneous forms W in $\mathcal{W} = R_{q_1}^{Nl_1} \times R_{q_2}^{Nl_2} \times \cdots \times R_{q_n}^{Nl_n}$, where generic means that W is a point of a Zariski open set of $\mathcal{W}$.

Proposition 2.4. *There exists a Zariski open set $\mathcal{U} \subseteq \mathcal{W}$ such that, for any $W \in \mathcal{U}$, I_W and I have the same graded Betti numbers.*

Proof. By virtue of Lemma 2.3 it is enough to show that the graded Betti numbers of I_W are the same as those of $I^{\mathbf{P}}$. The kernel of σ_W is generated by a $P/I^{\mathbf{P}}$-regular sequence if and only if $\operatorname{Tor}_m^P(P/\operatorname{Ker}\sigma_W, P/I^{\mathbf{P}}) = 0$ for all $m > 0$. This is an open property on $\mathcal{W}$. The Zariski open set $\mathcal{U}$ is not empty since $\{Z_{ijh} - Z_{ij1}\}$ is a $P/I^{\mathbf{P}}$-regular sequence (cf. the proof of the previous Lemma). Thus, if $W \in \mathcal{U}$, $\operatorname{Ker}\sigma_W$ is generated by a P- and $P/I^{\mathbf{P}}$-regular sequence. By Lemma 2.2, I_W has the same graded Betti numbers as $I^{\mathbf{P}}$. $\square$

As an important consequence for our purposes, we thus obtain that, if W is a generic collection of homogeneous forms, then I and I_W have the same Hilbert function. Macaulay's Theorem now yields that I and $\operatorname{in}(I_W)$ have the same Hilbert function. Moreover, since the Hilbert function of R/I_W

is minimal when $\operatorname{Ker}\sigma_W$ is generated by a $P/I^{\mathbf{P}}$-regular sequence, H_{I_W} is maximal when W is generic.

The lexicographic order on the set of monomials subspaces of R_d is defined as follows. If $\dim V > \dim W$ then $V >_{\text{lex}} W$. If V and W are spanned by $X^{\mu_1} >_{\text{lex}} \ldots >_{\text{lex}} X^{\mu_m}$ and $X^{\eta_1} >_{\text{lex}} \ldots >_{\text{lex}} X^{\eta_m}$ respectively, then $V >_{\text{lex}} W$ if there exists $s < m$ such that $X^{\mu_s} >_{\text{lex}} X^{\eta_s}$ and $X^{\mu_i} = X^{\eta_i}$ for every $i < s$. We can thus order the monomials of $\bigwedge^m R_d$ lexicographically.

Proposition 2.5. *Let W be a generic collection of forms of $\mathcal{W}$. Then, for all $d \geq 0$, $\operatorname{in}(I_W)_d$ is the greatest monomial subspace which can occur for any $W \in \mathcal{W}$.*

Proof. First observe that, if $I \subseteq (R, w)$ is a homogeneous ideal, $\operatorname{in}(I_d)$ is spanned by $X^{\mu_1}, \ldots, X^{\mu_m}$ and I_d is spanned by $g_1, \ldots, g_m$, then $\operatorname{in}(g_1 \wedge \cdots \wedge g_m) = X^{\mu_1} \wedge \cdots \wedge X^{\mu_m}$. In fact, after a change of basis, one may assume that $\operatorname{in}(g_i) = X^{\mu_i}$. Moreover, if I is a monomial ideal with $\dim I_d = m$, $H_{I_W}(d) \geq m$ for any $W \in \mathcal{W}$. Let $X^{\mu_1} \wedge \cdots \wedge X^{\mu_m}$ be the greatest monomial that ever occurs as $\operatorname{in}(\bigwedge^m (I_W)_d)$ for any W, then for a generic W

$$\operatorname{in}\left(\bigwedge^m (I_W)_d\right) = X^{\mu_1} \wedge \cdots \wedge X^{\mu_m}.$$

This is easily seen: the coefficient of $X^{\mu_1} \wedge \cdots \wedge X^{\mu_m}$ in $\bigwedge^m \sigma_W(I^{\mathbf{P}})_d$ is a polynomial in the coefficients of $\{f_{ijh}\}$; since $X^{\mu_1} \wedge \cdots \wedge X^{\mu_m}$ occurs as a monomial of $\bigwedge^m \sigma_W(I^{\mathbf{P}})_d$ for some W, it must occur for an open set $\mathcal{U}$ in $\mathcal{W}$. For each $W \in \mathcal{U}$, $X^{\mu_1} \wedge \cdots \wedge X^{\mu_m}$ is the initial term of $\bigwedge^m \sigma_W(I^{\mathbf{P}})_d$. Thus, $\operatorname{in}(I_W)_d = (X^{\mu_1}, \ldots, X^{\mu_m})$, as desired. □

After taking the initial ideal with respect to the lexicographic order we may assume that I is a monomial ideal. We let $\Phi(I) \doteq \operatorname{in}(I_W)$, where W is a generic collection of forms as above, and we denote the s-fold application of Φ by $\Phi^s(I)$. What we have shown above proves that $\Phi(I)$ is well-defined and has the same Hilbert function as I. Moreover, $\Phi(I)_d \geq_{\text{lex}} I_d$ for every d. As a consequence $\Phi(I) = I$ if I is a lexicographic ideal and there exists a minimum index s such that $\Phi^t(I) = \Phi^s(I)$ for any $t > s$. We say that the ideal $\Phi^s(I)$ is a *complete polarization* of the ideal I and we denote it by $I^{\mathbf{P}}$.

Examples 2.6. a) Let $(R, w) = (K[X, Y, Z, T, U], (1, 2, 2, 3, 3))$ and consider the ideal $I = (X^4, YT, X^2T, YZ^2)$. We can easily check that I is not lexifiable. The complete polarization of I is reached after three steps,

$$\begin{aligned}\Phi(I) = (&X^4, X^3Y, X^3Z, X^3T, X^2Y^2, X^2YZ, X^2YT, X^2ZT, X^2T^2, X^2Z^3,\\ &XY^2T, XYZT, XY^3Z, XY^4, XY^2Z^2, Y^4Z, Y^5, XYZ^4, XYT^3,\\ &Y^3T^2, YT^4);\end{aligned}$$

$$\begin{aligned}\Phi^2(I) = (&X^4, X^3Y, X^3Z, X^2Y^2, X^2YZ, X^3T, X^2YT, X^2ZT, XY^2T, XYZT,\\ &X^2Z^3, X^2T^2, XY^4, XY^3Z, XY^2Z^2, Y^5, Y^4Z, XYZ^4, Y^3T^2, XYT^3,\\ &XY^2U^3, YT^5, XYZ^3U^3, Y^2ZT^4, YZ^3T^4);\end{aligned}$$

$$\begin{aligned}I^{\mathbf{P}} = \Phi^3(I) = (&X^4, X^3Y, X^3Z, X^3T, X^2Y^2, X^2YZ, X^2YT, X^2Z^3, X^2ZT,\\ &X^2T^2, XY^4, XY^3Z, XY^2Z^2, XY^2T, XY^2U^3, XYZ^4, XYZ^3U^3,\\ &XYZT, XYZU^4, XYT^3, Y^5, Y^4Z, Y^3T^2, Y^2ZT^4, Y^2T^5,\\ &YZ^3T^4, YZ^2T^5, YT^6).\end{aligned}$$

b) Let $(R, w) = (K[X, Y, Z], (1, 2, 4))$, $I_1 = (Y^2, X^2Y, XYZ)$ and $I_2 = (X^3, Y^2)$. One verifies that I_1 and I_2 are lexifiable and ${I_1}^{\text{lex}} = I_1^{\mathbf{P}} = (X^4, X^3Z, X^2Y)$, whereas $I_2^{\mathbf{P}} = (X^3, X^2Y, XY^2, Y^3)$ and ${I_2}^{\text{lex}} = (X^3, X^2Y, X^2Z, XY^2, Y^4)$.

3 Regularity

Let M be a finitely generated graded module with $\operatorname{proj\,dim} M = r$ and let $b_i(M) \doteq \max_{j\in\mathbb{Z}}\{\beta_{ij}(M) \neq 0\}$, for $i = 0, \ldots, r$. In this section we provide a detailed proof of the following theorem.

Theorem 3.1 ([DS] Theorem 3.5)**.** *Let $R = K[X_1, \ldots, X_l]$ with* $\deg X_i = q_i$. *Let M be a finitely generated R-module. Then*

$$\operatorname{reg} M = \max_{i\geq 0}\{b_i(M) - i\} - \sum_{j=1}^{l}(q_j - 1).$$

Observe that local cohomology modules of a graded module over a weighted polynomial ring have a natural graded structure so that Castelnuovo-Mumford regularity can be still defined by means of local cohomology. In fact, if $H^i_{\mathfrak{m}}(M)$ denotes the i^{th} graded local cohomology module of the graded R-module M with support on the graded maximal ideal $\mathfrak{m}$ and we let $a^i(M)$ be $\max\{j \in \mathbb{Z}\colon H^i_{\mathfrak{m}}(M)_j \neq 0\}$ if $H^i_{\mathfrak{m}}(M) \neq 0$ and $-\infty$ otherwise, the *Castelnuovo-Mumford regularity* of M is defined, as usual, by $\operatorname{reg} M = \max_{0\leq i\leq \dim M}\{a^i(M) + i\}$. Notice also that, in case of a standard graduation, the second term on the right-hand side of the formula vanishes giving back the well-known characterization of regularity by means of graded Betti numbers.

Theorem 3.1 provides a method to compute the regularity of an (R, w)-module M without using Noether Normalization but directly from its minimal resolution as an (R, w)-module, as shown in the following easy example.

Example 3.2. Let $(R, w) = (K[X, Y, Z], (1, 2, 3))$ and $I = (Z^2 - X^6, Y^2 - X^4)$. Then R/I is 1-dimensional and $K[X]$ is a Noether Normalization, since both

$\overline{Y}$ and $\overline{Z}$ are integral over $K[X]$. Clearly, $\{\overline{1}, \overline{Y}, \overline{Z}, \overline{YZ}\}$ is a minimal system of generators of R/I as a $K[X]$-module and the first syzygy module is 0. Therefore a minimal graded resolution of R/I as a $K[X]$-module is

$$0 \to K[X] \oplus K[X](-2) \oplus K[X](-3) \oplus K[X](-5) \to R/I \to 0.$$

By Theorem 5.5 in [Be] we have that $\operatorname{reg} R/I = 5$, since $\deg X = 1$. On the other hand, a minimal graded resolution of R/I as an R-module is $0 \to R(-10) \to R(-4) \oplus R(-6) \to R/I \to 0$ and Theorem 3.1 yields $\operatorname{reg} R/I = 10 - 2 - (0 + 1 + 2) = 5$.

We thus have a tool for the calculation of regularity which is only based on Gröbner bases computations. This can be of some advantage, since to find a Noether Normalization may be quite unpleasant. In the standard case, a Noether Normalization can be obtained by choosing a collection of generic linear forms of length $\dim M$ (see for instance [V]). In a non-standard situation, the weighted counterpart of Prime Avoidance only grants that such generic forms can be chosen of degree q.

The following results descend easily from the basic properties of local cohomology.

Lemma 3.3. *Let $0 \to N \to M \to Q \to 0$ be a short exact sequence of finitely generated graded R-modules. Then we have:*

(i) $\operatorname{reg} N \le \max\{\operatorname{reg} M, \operatorname{reg} Q + 1\}$.

(ii) $\operatorname{reg} M \le \max\{\operatorname{reg} N, \operatorname{reg} Q\}$.

(iii) $\operatorname{reg} Q \le \max\{\operatorname{reg} N - 1, \operatorname{reg} M\}$.

(iv) If N has finite length, then $\operatorname{reg} M = \max\{\operatorname{reg} N, \operatorname{reg} Q\}$.

Proof. We start by proving (i). Consider the long exact sequence in cohomology $\ldots \to H^{i-1}_{\mathfrak{m}}(Q) \to H^i_{\mathfrak{m}}(N) \to H^i_{\mathfrak{m}}(M) \to \ldots$. Let $\alpha \doteq \max\{\operatorname{reg} M, \operatorname{reg} Q + 1\}$ and observe that $a^0(N) \le a^0(M) \le \operatorname{reg} M \le \alpha$, while $H^{i-1}_{\mathfrak{m}}(Q)_{\alpha-i+1} = 0$ for all $i \ge 1$, since $\alpha > \operatorname{reg} Q$. Thus, it is sufficient to verify that $a^i(N) \le \alpha - i$ for all $i \ge 1$, and this follows immediately from the fact that $H^i_{\mathfrak{m}}(N)_{\alpha-i+1} \simeq H^i_{\mathfrak{m}}(M)_{\alpha-i+1} = 0$, for all $i \ge 1$. The proofs of (ii) and (iii) are similar. As for the proof of (iv), it is clear that $\operatorname{reg} N = a^0(N)$ and $a^0(M)$ equals $\max\{a^0(N), a^0(Q)\}$. Thus, $\operatorname{reg} M \doteq \max\big\{a^0(M), \max_{i>0}\{a^i(M) + i\}\big\}$, which is $\max\big\{a^0(N), a^0(Q), \max_{i>0}\{a^i(Q) + i\}\big\}$ and we are done. ☐

As an application one gets that, if M is a finitely generated graded R-module and $x \in R_d$ is non-zerodivisor on M, then $\operatorname{reg} M/xM = \operatorname{reg} M+(d-1)$. More generally, if x is such that $(0:_M x)$ has finite length, then

$$\operatorname{reg} M = \max\{\operatorname{reg} 0:_M x, \operatorname{reg} M/xM - (d-1)\}.$$

This is easily seen considering the exact sequence

$$0 \to (0:_M x)(-d) \to M(-d) \to M \to M/xM \to 0$$

and splitting it into the two short exact sequences

$$\begin{aligned} 0 \to &(0:_M x)(-d) \to M(-d) \to xM \to 0 \\ &0 \to xM \to M \to M/xM \to 0. \end{aligned} \tag{3.1}$$

We need now two more preparatory results.

Lemma 3.4. *Let $x \in R_d$, $d > 0$, such that $0:_M x$ is of finite length. Then, for all $i \geq 0$,*

$$a^{i+1}(M) + d \leq a^i(M/xM) \leq \max\{a^i(M), a^{i+1}(M) + d\}.$$

Proof. From (3.1) we deduce that $H^i_{\mathfrak{m}}(M(-d)) \simeq H^i_{\mathfrak{m}}(xM)$ for all $i > 0$, and, therefore, $a^i(xM) = a^i(M) + d$ for all $i > 0$. If $a^i(M/xM)$ were smaller than $a^{i+1}(M) + d$, from the the long exact sequence in cohomology

$$\ldots \to H^i_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(M/xM) \to H^{i+1}_{\mathfrak{m}}(xM) \to H^{i+1}_{\mathfrak{m}}(M) \to \ldots$$

in degree $\alpha \doteq a^{i+1}(M)+d$, one would have $0 = H^i_{\mathfrak{m}}(M/xM)_\alpha \to H^{i+1}_{\mathfrak{m}}(xM)_\alpha \to H^{i+1}(M)_\alpha = 0$, which is a contradiction since the middle term is not equal to 0. This completes the proof of the first inequality. The second inequality can be proven in a similar way. $\square$

Lemma 3.5. *With the above notation, $b_0(M) \leq \operatorname{reg} M + \sum_{j=1}^{l}(q_j - 1)$.*

Proof. Using downward induction on s, we prove that

$$b_0(M/(X_1, \ldots, X_s)M) \leq \max_{i \geq 0}\{a^i(M/(X_1, \ldots, X_s)M) + i\} + \sum_{j=s+1}^{r}(q_j - 1).$$

If $s = l$ then $M/(X_1, \ldots, X_l)M$ is Artinian and it coincides with its 0^{th} local cohomology, whereas its higher local cohomology modules vanish. Thus $a^0(M/(X_1, \ldots, X_l)M)$ is the highest degree of an element in the module itself and it is obviously bigger than $b_0(M/(X_1, \ldots, X_l)M)$.

For the sake of notational simplicity, let $N \doteq M/(X_1, \dots, X_s)$. Suppose that the above displayed equation holds true for $N/X_{s+1}N$. An application of Nakayama's Lemma provides $b_0(N) = b_0(N/X_{s+1}N)$; hence, the inductive hypothesis yields

$$\begin{aligned} b_0(N) &\le \max\{a^0(N),\ b_0(N/X_{s+1}N)\} \\ &\le \max\Big\{a^0(N) + \sum_{j=s+1}^{l}(q_j - 1),\ b_0(N/X_{s+1}N)\Big\} \\ &\le \max\Big\{a^0(N) + \sum_{j=s+1}^{l}(q_j - 1),\ \max_{i\ge 0}\{a^i(N/X_{s+1}N) + i\} + \sum_{j=s+2}^{r}(q_j - 1)\Big\} \\ &= \max\Big\{a^0(N),\ \max_{i\ge 0}\{a^i(N/X_{s+1}N) + i + 1 - q_{s+1}\}\Big\} + \sum_{j=s+1}^{r}(q_j - 1). \end{aligned}$$

By virtue of the previous Lemma,

$$\begin{aligned} &\max\Big\{a^0(N),\ \max_{i\ge 0}\{a^i(N/X_{s+1}N) + i + 1 - q_{s+1}\}\Big\} \\ &\le \max\Big\{a^0(N),\ \max_{i\ge 0}\{a^i(N) + i + 1 - q_{s+1}, a^{i+1}(N) + q_{s+1} + i + 1 - q_{s+1}\}\Big\} \\ &= \max_{i\ge 0}\{a^i(N) + i\} = \operatorname{reg} N, \end{aligned}$$

since $1 - q_{s+1} \le 0$, and this completes the proof. □

Proof of Theorem 3.1. We prove the assertion by induction on the projective dimension of M. If $\operatorname{proj\,dim} M = 0$, then M is a free module and its regularity equals $\max_{i\ge 0}\{a^i(M) + i\} = a^l(M) + l$. If $M = R$ then, by Local Duality,

$$\begin{aligned} a^l(R) &= \max\{j \in \mathbb{Z}\colon H^l_{\mathfrak{m}}(R)_j \ne 0\} = -\min\{j \in \mathbb{Z}\colon \operatorname{Hom}_R(R, \omega_R)_j \ne 0\} \\ &= -\min\Big\{j \in \mathbb{Z}\colon \operatorname{Hom}_R\big(R, R(-\sum_{h=1}^{l} q_h)\big)_j \ne 0\Big\} \\ &= -\min\Big\{j \in \mathbb{Z}\colon R\big(-\sum_{h=1}^{l} q_h\big)_j \ne 0\Big\} = -\sum_{h=1}^{l} q_h. \end{aligned}$$

For an arbitrary finitely generated free graded R-module $M = \oplus R(-c)$, since local cohomology is additive, $a^l(M)$ equals the largest $a^l(R(-c))$, which is clearly $a^l(R(-b_0(M)))$. Thus,

$$\operatorname{reg} M = a^l(M) + l = b_0(M) - \sum_{j=1}^{l}(q_j - 1).$$

We may now assume that $\operatorname{proj}\dim M \geq 1$. If we let $0 \to N \to F \to M \to 0$ be the first step of a minimal graded free resolution of M, we immediately see that $b_0(F) = b_0(M)$ and $b_i(N) = b_{i+1}(M)$. Since $H^i_{\mathfrak{m}}(F) = 0$ for $i \neq l$ and $a^l(F) = b_0(M) - \sum_{j=1}^l q_j$, the long exact sequence in cohomology $\ldots \to H^{i-1}_{\mathfrak{m}}(N) \to H^{i-1}_{\mathfrak{m}}(F) \to H^{i-1}_{\mathfrak{m}}(M) \to H^i_{\mathfrak{m}}(N) \to \ldots$ shows that

$$a^0(N) = -\infty \quad \text{and} \quad a^i(N) = a^{i-1}(M) \text{ for all } 0 < i < l.$$

Moreover, from the exact sequence $0 \to H^{l-1}_{\mathfrak{m}}(M) \to H^l_{\mathfrak{m}}(N) \to H^l_{\mathfrak{m}}(F) \to H^l_{\mathfrak{m}}(M) \to 0$, it is easy to see that

$$a^l(M) \leq a^l(F) \quad \text{and} \quad a^{l-1}(M) \leq a^l(N) \leq \max\{a^{l-1}(M), a^l(F)\}.$$

Therefore

$$\begin{aligned} \operatorname{reg} M = \max_{i\geq 0}\{a^i(M) + i\} &\leq \max\bigl\{\max_{i\geq 0}\{a^i(N) + i - 1\}, a^l(M) + l\bigr\} \\ &\leq \max\Bigl\{\max_{i\geq 0}\{a^i(N) + i - 1\}, b_0(M) - \sum_{j=1}^l q_j + l\Bigr\}. \end{aligned} \tag{3.2}$$

By Lemma 3.5, $\operatorname{reg} M \geq b_0(M) - \sum_{j=1}^l q_j + l$, which implies that the inequalities in (3.2) are equalities. Now we can make use of the inductive assumption on N and obtain

$$\begin{aligned} \operatorname{reg} M &= \max\bigl\{\operatorname{reg} N - 1,\ b_0(M) - \sum_{j=1}^l q_j + l\bigr\} \\ &= \max\Bigl\{\max_{i\geq 0}\{b_i(N) - i - 1\} - \sum_{j=1}^l (q_j - 1),\ b_0(M) - \sum_{j=1}^l (q_j - 1)\Bigr\} \\ &= \max\bigl\{\max_{i>0}\{b_{i+1}(M) - i - 1\},\ b_0(M)\bigr\} - \sum_{j=1}^l (q_j - 1) \\ &= \max_{i\geq 0}\{b_i(M) - i\} - \sum_{j=1}^l (q_j - 1), \end{aligned}$$

as desired. □

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