An. Şt. Univ. Ovidius Constanţa Vol. 9(1), 2001, 101–106

ON EQUATIONS IN BOUNDED LATTICES

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Abstract

The following properties of Boolean equations are well known: every system of equations is equivalent to a single equation of the form

$$f(x_1, \ldots, x_n) = 1,$$

the consistency condition for such an equation, the method of successive elimination of variables, and the formula for the general (reproductive) solution using a particular solution. These features are shared by Post equations and by equations in functionally complete algebras. In this paper we extend the above results to bounded lattices endowed with a supplementary binary operation (the Kronecker delta). As a by-product we obtain a generalization of the concept of functionally complete algebra, by dropping the finiteness assumption.

The following important properties of Boolean equations are well known (see e.g. [12]): 1) every system of equations is equivalent to a single equation of the form $f(x_1, \ldots, x_n) = 1$; 2) the consistency condition for such an equation; 3) the solution by the method of successive elimination of variables, and 4) the parametric formula for the set of solutions, using a particular solution. These feature have beeen extended to Post equations by Carvallo [5]–[7], Serfati [14]–[16], Beazer [1] and Bordat [2], [3], while Nipkow [8] proved that the same properties hold in functionally complete algebras, and obtained further generalizations [9]; see also [13]. In this mainly expository note we apply Nipkow's technique to obtain the same results in a bounded lattice endowed with the Kronecker delta as a supplementary operation.

Recall that an *algebra* is a set A equipped with a family of finitary operations. By a *polynomial* over A is meant a function $f : A^n \longrightarrow A$, $n \in \mathbf{N}$, which has an expression built up from variables and the basic operations of A. A function $f : A^n \longrightarrow A$, $n \in \mathbf{N}$, is said to be *algebraic* provided it can be

Received: August, 2001.

obtained from a polynomial by replacing certain (possibly none) apparitions of variables by constants of A. By a *functionally complete* (*primal*) algebra is meant a finite algebra A such that every function $f : A^n \longrightarrow A$, $n \in \mathrm{N}$ N , is algebraic (a polynomial). Post [10] proved that a finite algebra A is functionally complete if and only if there exist two distinct elements $0, 1 \in A$ and two algebraic functions $+$ and $\cdot$ on A such that for every $x \in A$,

$$x + 0 = 0 + x = x \ , \tag{1}$$

$$x \cdot 0 = 0 \cdot x = 0 \ , \tag{2}$$

$$x \cdot 1 = x \ , \tag{3}$$

and for every $a \in A$, the Kronecker function $\delta_a : A \longrightarrow A$ defined by

$$\delta_a(x) = 1 \text{ if } x = a \ , \text{ else } 0 \ , \tag{4}$$

is algebraic. The sufficiency of these conditions was rediscovered 50 years later by Prešić [11]. See also [13], Propositions 13.2.1 and 1.2.3.

Nipkow [8] proved that the above properties 1) – 4) hold for equations in a functionally complete algebra and gave applications to equations in the Post algebra $C_r = \{0, 1, \ldots, r-1\}$ and beyond lattice theory, to matrix rings. He then extended the results to direct powers of primal algebras and to varieties generated by primal algebras, with examples in certain 3–rings [9]. Büttner [4] suggested a promising approach to solving arbitrary equations (i.e., not necessarily expressed by algebraic functions) over a finite algebra. Namely, the signature of the algebra is enriched so as to obtain a functionally complete algebra, and the original equation becomes an algebraic equation which is solved by unification theory techniques.

The present note may be viewed as an application of Nipkow's technique via Büttner's idea. We start from the striking fact that the join and meet operations of a bounded lattice satisfy conditions (1)–(3). The next point is the remark that if we enrich the lattice structure by adding the Kronecker delta , then every function defined on the new algebra is algebraic, although the underlying set need not be finite. We thus obtain a generalization of the concept of functionally complete algebra in which Nipkow's technique works and therefore we recapture properties 1) – 4) within this framework.

Given a bounded lattice $(L; \wedge, \vee, 0, 1)$, let $\bigwedge_{i \in I} x_i$ and $\bigvee_{i \in I} x_i$ denote the infimum and supremum of an arbitrary subset $\{x_i \mid i \in I\}$ of L, whenever these elements exist. Further, let $\delta : L^2 \longrightarrow L$ be defined by

$$\delta(x, y) = 1 \text{ if } x = y \ , \text{ else } 0 \ . \tag{5}$$

Bearing in mind the Post theorem mentioned above, we may be tempted to introduce the family of unary Kronecker deltas (4), for $a \in L$, as new operations. However, since

(6) $$\delta_a(x) = \delta(a, x) \ ,$$

(7) $$\delta(x, y) = \bigvee_{a \in L} \delta_a(x) \wedge \delta_a(y) \ ,$$

taking the binary Kronecker delta (5) as the new operation is likely to provide a simpler approach. Therefore, in the following we will work in the algebra

(8) $$\mathbf{L} = (L; \wedge, \vee, \delta, 0, 1) \ .$$

Proposition 1 *For every $n \in \mathbf{N}$, every function $f : L^n \longrightarrow L$ is algebraic.*

Proof: It suffices to prove the identity

(9) $$f(x_1, \ldots, x_n) = \bigvee_{(a_1,\ldots,a_n) \in L^n} f(a_1, \ldots, a_n) \wedge \delta(a_1, x_1) \wedge \ldots \delta(a_n, x_n) \ .$$

But for every $(x_1, \ldots, x_n) \in L^n$, the right side of (9) exists and equals

$$f(x_1, \ldots, x_n) \wedge \delta(x_1, x_1) \wedge \cdots \wedge \delta(x_n, x_n) = f(x_1, \ldots, x_n) \ .$$

□

Proposition 2 *Every system of equations of the form*

(10.1) $$g_i(x_1, \ldots, x_n) = h_i(x_1, \ldots, x_n) \qquad (i \in I) \ ,$$

(10.2) $$g_j(x_1, \ldots, x_n) \leq h_j(x_1, \ldots, x_n) \qquad (j \in J) \ ,$$

(10.3) $$g_k(x_1, \ldots, x_n) \neq h_k(x_1, \ldots, x_n) \qquad (k \in K) \ ,$$

where $I \cup J \cup K$ is a finite non-empty set, is equivalent to a single equation of the form

(11) $$f(x_1, \ldots, x_n) = 1 \ .$$

Proof: Each equation (10.1) can be written in the form

$$\delta(g_i(x_1, \ldots, x_n), h_i(x_1, \ldots, x_n)) = 1$$

and a similar result holds for each inequality (10.2), because it can be written in the form $g_j = g_j \wedge h_j$. Further, each non-equation (10.3) can be written in the form

$$\delta(\delta(g_k(x_1, \ldots, x_n), h_k(x_1, \ldots, x_n)), 0) = 1 \ .$$

Finally, a system of the form

$$f_r(x_1, \ldots, x_n) = 1 \qquad (r \in I \cup J \cup K)$$

is equivalent to the single equation

$$\bigwedge_{r \in I \cup J \cup K} f_r(x_1, \ldots, x_n) = 1 \ .$$

□

Proposition 3 *Equation* (11) *is consistent if and only if*

$$\bigvee_{(a_1,\ldots,a_n) \in L^n} \delta(f(a_1, \ldots, a_n), 1) = 1 \ . \tag{12}$$

Proof: The left side of (12) exists and equals 1 if and only if there is $(a_1, \ldots, a_n) \in L^n$ such that $f(a_1, \ldots, a_n) = 1$. □

The next proposition may be regarded as the basis for the method of successive elimination of variables.

Proposition 4 A) *Equation* (11) *is consistent if and only if the equation in* $n-1$ *unknowns*

$$\bigvee_{a \in L} \delta(f(x_1, \ldots, x_{n-1}, a), 1) = 1 \tag{13}$$

is consistent.

B) *When this is the case, a vector* $(a_1, \ldots, a_n) \in L^n$ *is a solution of* (11) *if and only if* $(a_1, \ldots, a_{n-1})$ *satisfies* (13)*, while* a_n *is a solution of the equation*

$$f(a_1, \ldots, a_{n-1}, x) = 1 \ . \tag{14}$$

Proof: Remark that it suffices to prove B).

Suppose $(a_1, \ldots, a_n)$ is a solution of equation (11). Then a_n satisfies equation (14), therefore

$$\bigvee_{a \in L} \delta(f(a_1, \ldots, a_{n-1}, a), 1) = 1 \tag{15}$$

by Proposition 3 applied with $n := 1$. Condition (15) shows that $(a_1, \ldots, a_{n-1})$ is a solution of equation (13).

Conversely, suppose $(a_1, \ldots, a_{n-1})$ satisfies equation (13). Then (15) holds, therefore equation (14) is consistent, again by Proposition 3 with $n := 1$. For every solution a_n of equation (14) we have $f(a_1, \ldots, a_{n-1}, a_n) = 1$. □

Now recall a very general definition which applies in particular to equation (11). Consider a vector $(\varphi_1, \ldots, \varphi_n)$, where $\varphi_i : L^n \longrightarrow L$ $(i = 1, \ldots, n)$. Formulas

$$x_i = \varphi_i(p_1, \ldots, p_n) \qquad (i = 1, \ldots, n) \tag{16}$$

define the *reproductive general solution* of equation (11) provided

$$f(\varphi_1(p_1, \ldots, p_n), \ldots, \varphi_n(p_1, \ldots, p_n)) = 1 \qquad (\forall p_1, \ldots, p_n \in L) \tag{17}$$

and every solution $(x_1, \ldots, x_n)$ of equation (11) satisfies

(18) $$x_i = \varphi_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) .$$

Proposition 5 *Suppose* $(a_1, \ldots, a_n)$ *is a solution of equation* (11). *Then formulas*

(19) $$x_i = p_i\delta(f(p_1, \ldots, p_n), 1) \vee a_i\delta(\delta(f(p_1, \ldots, p_n), 1), 0)$$
$$(i = 1, \ldots, n)$$

define the reproductive general solution of equation (11).

PROOF: Denote the right sides of formulas (19) by φ_i. Take $(p_1, \ldots, p_n) \in L$. If $f(p_1, \ldots, p_n) = 1$ then

$$\varphi_i(p_1, \ldots, p_n) = p_i \qquad (i = 1, \ldots, n) ,$$

i.e., relations (18) hold, and

$$f(\varphi_1(p_1, \ldots, p_n), \ldots, \varphi_n(p_1, \ldots, p_n)) = f(p_1, \ldots, p_n) = 1 .$$

Otherwise

$$\varphi_i(p_1, \ldots, p_n) = a_i \qquad (i = (1, \ldots, n) ,$$

hence

$$f(\varphi_1(p_1, \ldots, p_n), \ldots, \varphi_n(p_1, \ldots, p_n)) = f(a_1, \ldots, a_n) = 1 .$$

□

Conclusions The technique used in this note is borrowed from [8], but the framework of a bounded lattice provides simpler proofs and results, including "functional completeness" without the finiteness assumption.

As a matter of fact, this framework can easily be implemented on a quite arbitrary set of cardinality ≥ 3. For such a set can be made into a flat lattice, that is, a bounded lattice in which the elements $\neq 0, 1$ are pairwise uncomparable.

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