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INDICATIVE PROPOSITIONS

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Abstract

Two new problems of bivalent propositional logic are proposed here: firstly, to distinguish the sense of propositions, besides the logical value and secondly, to analyze the "ponderal" difference between two parts of a proposition: subject-predicate.

1 Relational projections and extensions

Let $r = (A_1, A_2, A_3, R)$ be a ternary relation (see [3]). Starting with this, we may define three binary relations (induced projections), namely $r_{12} = (A_1, A_2, R_{12})$, $r_{23} = (A_2, A_3, R_{23})$ and $r_{13} = (A_1, A_3, R_{13})$ defined by

$$(x_1, x_2) \in R_{12} \iff \text{ there exists } x_3 \in A_3 \text{ such that } (x_1, x_2, x_3) \in R$$

and its analogs.

Denote

$$r < x_1, x_2 >= \{x_3 \in A_3 \mid (x_1, x_2, x_3) \in R\}$$

and analogously $r < x_2, x_3 >$ and $r < x_1, x_3 >$, where $(x_1, x_2, x_3) \in A_1 \times A_2 \times A_3$.

Proposition 1.1 $x_i \, r_{ij} \, x_j \iff r < x_i, x_j > \neq \emptyset$, *where* $i, j \in \{1, 2, 3\}$, $i < j$.

This proposition follows by the above definitions.

To each of the binary relations r_{ij}, $i, j \in \{1, 2, 3\}$, $i < j$, we associate a ternary *ponderal extension* $\widetilde{r}_{ij} = (A_i, A_j, \mathbb{N}, \widetilde{R}_{ij})$ defined by

$$(x_1, x_2, n) \in \widetilde{R}_{12} \iff (x_1, x_2) \in R_{12} \text{ and } |r < x_1, x_2 > | = n$$

and its analogs.

Remark. The relational projections and extensions may be generalized to the case of arbitrary n-ary relations.

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2 Logical interpretation

In the case of a bivalent propositional logic $(\mathbf{P}, v)$ (see [2]), we imagine a generic proposition "S is P" (S-subject, P-predicate).

For an algebraic formulation, we consider the set M of individuals and the set Π of predicative letters such that for each $\mathcal{P} \in \Pi$ it is defined a function

$$\mathcal{P} : M \to \mathbf{P}$$

which associates to every $x \in M$ the proposition $\mathcal{P}(x)$ with the signification "x has the property $\mathcal{P}$". Therefore, by the correspondence $S \mapsto x$, $P \mapsto \mathcal{P}$, the proposition "S is P" receives the algebrized form $\mathcal{P}(x)$.

Since the function $\mathcal{P} : M \to \mathbf{P}$ may be extended by the bivalent valuation $v : \mathbf{P} \to V = \{0, 1\}$ to

$$v \circ \mathcal{P} : M \to V\,,$$

we must accept that for each $x \in M$, $v(\mathcal{P}(x)) \in \{0, 1\}$, that is the proposition $\mathcal{P}(x)$ is either false or true.

In this way, the following problem arises: does the proposition $\mathcal{P}(x)$ make sense for every $\mathcal{P} \in \Pi$ and $x \in M$?

Example 2.1 Let $\mathcal{P}, \mathcal{Q}$ be predicative letters with the significations:

$$\mathcal{P}(x) = \text{"}x \text{ is round "}\,,$$

$$\mathcal{Q}(x) = \text{"}x \text{ is nervous "}\,.$$

For $x =$"square", $\mathcal{P}(x)$ does make sense (and it is false), but $\mathcal{Q}(x)$ does not.

Starting with the above example, we seek for an algebraic definition for the notion of "sense".

First consider that the proposition $\mathcal{P}(x)$ does make sense if there exists a method to establish the truth-value $v(\mathcal{P}(x)) \in \{0, 1\}$. But for this purpose, we must suppose the existence of an "individual receiver", able to effect the valuation. Consequently, we consider suitable the following algebraic definition:

Definition 2.2 Let $r = (\Pi, M, \mathcal{R}, R)$ be a ternary relation, where Π and M have the above significations and $\mathcal{R}$ is the set of "individual receivers". The proposition $\mathcal{P}(x)$ ($\mathcal{P} \in \Pi$, $x \in M$) does make *sense* if $\mathcal{P}\, r_{12}\, x$ holds, where r_{12} is the first induced projection (see Section 1).

Proposition 2.3 *$\mathcal{P}(x)$ does make sense if and only if*

$$r < \mathcal{P}, x > \neq \emptyset\,.$$

(see Proposition 1.1).

Interpretation. The sense of a proposition consists of the existence of its individual receivers.

Remark. As a proposition assumes a communication, the notion of sense requires the existence of at least two individual receivers. This fact suggests a starker definition of sense, namely:

Definition 2.4 The proposition $\mathcal{P}(x)$ does make a *communicative sense* if

$$|r < \mathcal{P}, x > | \geq 2 \, .$$

Suggestion 1. The notion of n-*communicative sense* may be defined by the condition

$$|r < \mathcal{P}, x > | = n \, .$$

This definition is connected with the notion of ponderal extension (see Section 1).

We formulate now the following definition:

Definition 2.5 For a $\mathcal{P} \in \Pi$ and an $x \in M$, the *contextual universe* of $\mathcal{P}$ and of x is

$$\overset{\circ}{x}_{\mathcal{P}} = \{ x \in M \mid \mathcal{P}(x) \text{ does make sense } \}$$

and

$$\overset{\circ}{\mathcal{P}}_x = \{ \mathcal{P} \in \Pi \mid \mathcal{P}(x) \text{ does make sense } \}$$

respectively.

Proposition 2.6

$$x \in \overset{\circ}{x}_{\mathcal{P}} \Longleftrightarrow \mathcal{P} \in \overset{\circ}{\mathcal{P}}_x \, .$$

This fact follows by the following equalities:

$$\overset{\circ}{x}_{\mathcal{P}} = r_{12} < \mathcal{P} > \, , \quad \overset{\circ}{\mathcal{P}}_x = \overset{-1}{r}_{12} < x > \ .$$

Suggestion 2. We may imagine a three-valent logic, starting with the valuation

$$w : \mathbf{P} \to W = \{0, 1/2, 1\} \, ,$$

where

$$\mathbf{P} = \{ \mathcal{P}(x) \mid \mathcal{P} \in \Pi, x \in M \} \, ,$$

such that

$$\begin{cases} w(\mathcal{P}(x)) \in V = \{0, 1\} \, , & \text{if } \mathcal{P}(x) \text{ does make sense} \\ w(\mathcal{P}(x)) = 1/2 \, , & \text{otherwise} \end{cases}$$

3 Ponderal propositions

Each of the propositions $\mathcal{P}(x)$ determines the two contextual universes $\overset{\circ}{x}_{\mathcal{P}}$ and $\overset{\circ}{\mathcal{P}}_x$. These universes contain essential information, namely when the truth-value of the proposition $\mathcal{P}(x)$ depends only on one of the components.

Example 3.1 Given a Cramer system (S_n) of order n over $\mathbb{R}$, the proposition $\mathcal{P}(x)$ with the signification "x is a solution of (S_n)" has the contextual universe

$$\overset{\circ}{x}_{\mathcal{P}} = \mathbb{R}^n \,.$$

As (S_n) has a unique solution, to solve the system (S_n) consists of determining the (unique) solution $x_0 \in \mathbb{R}^n$, that is, to answer the question: "what is the solution of (S_n)?", with the proposition "x_0 is the solution of (S_n)".

Notice that the question is not if "there exists a solution", but "what is the solution?". So we are in a situation to put the accent on the subject (x_0).

Associate to the "global" proposition $\mathcal{P}(x)$, where $\mathcal{P} \in \overset{\circ}{\mathcal{P}}_x \subseteq \Pi$, $x \in \overset{\circ}{x}_{\mathcal{P}} = M$, the following two ponderal propositions:

$$(\widehat{x}) \quad \exists! x \in \overset{\circ}{x}_{\mathcal{P}} : \mathcal{P}(x) \,,$$

$$(\widehat{\mathcal{P}}) \quad \exists! \mathcal{P} \in \overset{\circ}{\mathcal{P}}_x : \mathcal{P}(x) \,.$$

where the symbol $\exists!$ denotes existence and uniqueness.

Denote by $\mathcal{P}(\widehat{x})$ and $\widehat{\mathcal{P}}(x)$ the two ponderal propositions respectively, namely:

$$\mathcal{P}(\widehat{x}) = \text{subject-ponderal}$$

$$\widehat{\mathcal{P}}(x) = \text{predicate-ponderal} \,.$$

Example 3.2 The "global" proposition $\mathcal{P}(x)$ with the signification "x is a perfect square" has the contextual universes:

$$\overset{\circ}{x}_{\mathcal{P}} = \mathbb{Z} \,,$$

$$\overset{\circ}{\mathcal{P}}_x = \{\text{all the numerical predicates}\} \,.$$

Although the global $\mathcal{P}(4)$ is true (4 is a perfect square), both of the ponderals $(\mathcal{P}(\widehat{4})$ and $\widehat{\mathcal{P}}(4))$ are false.

The example suggests the necessity to restrict the contextual universes. So take the restricted domains:

$$\widehat{\overset{\circ}{x}_{\mathcal{P}}} = \{2, 3, 4\} \subseteq \overset{\circ}{x}_{\mathcal{P}} = \mathbb{Z} \,,$$

$$\widehat{\overset{\circ}{\mathcal{P}}}_x = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\} \subseteq \overset{\circ}{\mathcal{P}}_x$$

with the significations:

$$\mathcal{P}_1(x) = \text{"}x \text{ is a prime"}$$
$$\mathcal{P}_2(x) = \text{"}x \text{ is a perfect square"}$$
$$\mathcal{P}_3(x) = \text{"}x \text{ is an odd number"}\,.$$

On these restricted domains, both of the ponderal propositions ($\mathcal{P}(\widehat{4})$ and $\widehat{\mathcal{P}}(4)$) are true.

Theorem 3.3 *Given the global $\mathcal{P}(x)$ and the corresponding ponderal propositions $\mathcal{P}(\widehat{x})$ and $\widehat{\mathcal{P}}(x)$, if one of the ponderals is true, then the global is also true.*

Proof. An equivalent formulation of the subject-ponderal is:

$$(\widehat{x}) \quad \mathcal{P}(x) \wedge \forall y(y \in \overset{\circ}{x}_{\mathcal{P}} \wedge y \neq x \supset \overline{\mathcal{P}(y)})\,.$$

Denote $\Phi(x) = \forall y(y \in \overset{\circ}{x}_{\mathcal{P}} \wedge y \neq x \supset \overline{\mathcal{P}(y)})$, as the variable y is bounded. So the definition $(\widehat{x})$ may be formulated by

$$(\widehat{x}) \quad \mathcal{P}(x) \wedge \Phi(x)\,.$$

Therefore, the first part of the theorem follows by the predicative identity:

$$\mathcal{P}(x) \wedge \Phi(x) \supset \mathcal{P}(x)\,.$$

Starting with the definition $(\widehat{\mathcal{P}})$, the second part of the theorem follows in a similar way.

Theorem 3.4 *If the global $\mathcal{P}(x)$ is true, then there exist the restricted domains $\widehat{\overset{\circ}{x}_{\mathcal{P}}} \subseteq \overset{\circ}{x}_{\mathcal{P}}$ and $\widehat{\overset{\circ}{\mathcal{P}}}_x \subseteq \overset{\circ}{\mathcal{P}}_x$, on which the ponderals $\mathcal{P}(\widehat{x})$ and $\widehat{\mathcal{P}}(x)$ are true.*

References

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