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GORENSTEIN TILED ORDERS WITH HEREDITARY RING OF MULTIPLIERS OF JACOBSON RADICAL *

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Abstract

The present paper is devoted to the description of Gorenstein tiled orders with hereditary ring of multipliers of the Jacobson radical. It is proved that all Gorenstein $(0, 1)$-tiled orders satisfy this property.

On the Algebraic Seminar of the Kiev Taras Shevchenko University Yu. Drozd had proposed a problem of description of Gorenstein tiled orders over discrete valuation ring (d.v.r.) with the hereditary ring of multipliers of the Jacobson radical. In our terminology a tiled order over a disrete valuation ring coincides with a prime semimaximal ring [ZK1], [ZK2].
Every tiled order Λ over a d.v.r. $\mathcal{O}$ is defined by the exponent matrix $\mathcal{E}(\Lambda)$. Hence for the fixed d.v.r. $\mathcal{O}$ it is sufficient to describe the exponent matrices of such orders. Moreover, we can consider a tiled order Λ be reduced, i.e. the exponent matrix $\mathcal{E}(\Lambda)$ have not symmetric zeroes. The reader is referred to [Sim 1] and [ZK1], [ZK2] for information on tiled orders.

I. Main result

Denote by $M_n(B)$ the ring of all square matrices of order n over a ring B. Let $\mathcal{E} \in M_n(\mathbf{Z})$. We shall call the matrix $\mathcal{E} = (\alpha_{ij})$ the exponent matrix if $\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}$ for $i, j, k = 1, \ldots, n$ and $\alpha_{ii} = 0$ for $i = 1, \ldots, n$. A matrix $\mathcal{E}$ is called a reduced exponent matrix if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$.

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Let $\mathcal{O}$ be a discrete valuation ring with the division ring of fractions T and $\mathcal{M} = \pi\mathcal{O} = \mathcal{O}\pi$ be an unique maximal ideal of $\mathcal{O}$. We shall build the tiled order in $M_n(T)$ by d.v.r. $\mathcal{O}$ and the exponent matrix $\mathcal{E} = (\alpha_{ij})$ of the following form:

$$\Lambda = \sum_{i,j=1}^{n} e_{ij}\pi^{\alpha_{ij}}\mathcal{O} \tag{1}$$

where e_{ij} are matrices units of $M_n(T)$. Obviously, $M_n(T)$ is both the left and right classical ring of fractions of Λ. We shall use the notation: $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$, where $\mathcal{E} = \mathcal{E}(\Lambda)$. A tiled order Λ is reduced if and only if $\alpha_{ij} + \alpha_{ji} > 0$ for $i, j = 1, \ldots, n$. This is equivalent to the fact that among the modules $e_{ii}\Lambda$ there are no isomorphic one; i.e. the quotient of Λ by Jacobson radical R is a direct product of division rings. Since any tiled order Λ is Morita equivalent to a reduced order, we shall restrict ourselves to reduced tiled orders.
Any two-sided ideal $J \subset \Lambda$ has the form:

$$J = \sum_{i,j=1}^{n} e_{ij}\pi^{\gamma_{ij}}\mathcal{O}$$

The matrix $\mathcal{E}(J) = (\gamma_{ij})$ will be called the exponent matrix of the ideal J. Recall [Z] that two reduced tiled orders over d.v.r. $\mathcal{O}$ in $M_n(T)$ are isomorphic if and only if their exponent matrices can be obtained one from another by elementary transformations of the following two types:

(1) subtracting the integer α from ith row with simultaneously adding it to ith column;

(2) simultaneously interchanging of two different rows and columns which have the same numbers.

Theorem 1.1. [K]. *The following conditions for a reduced tiled order* $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda) = (\alpha_{ij})\}$ *are equivalent:*

(a) Λ *is a Gorenstein order;*

(b) there exists a permutation $\sigma = \{i \to \sigma(i)\}$ *such that* $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ *for* $i, k = 1, \ldots, n$.

Definition. A Gorenstein tiled order Λ is called cyclic if a permutation σ is a cycle.

Now we consider the following reduced Gorenstein tiled orders:

(a) $H_m = H_m(\mathcal{O}) = \{\mathcal{O}, \mathcal{E}(H_m(\mathcal{O})) = (\alpha_{ij})\}$, where $\alpha_{ij} = 0$ for $i \le j$ and $\alpha_{ij} = 1$ for $i > j$; R_m is its Jacobson radical.

Obviously, H_m is a cyclic Gorenstein tiled order with the permutation $\sigma = (1\, m\, m-1 \ldots 3\, 2)$. Let $P_i = e_{ii}H_m$, $[P_i] = (1\,1 \ldots 1\,0\,0 \ldots 0)$ and $[\pi P_i] = (2\,2 \ldots 2\,1\,1 \ldots 1)$ for $i = 1, 2, \ldots, m$. Analogously, $Q_i = H_m e_{ii}$, $[Q_i] = (0\,0 \ldots 0\,1\,1 \ldots 1)^T$ and $[Q_i\pi] = (1\,1 \ldots 1\,2\,2 \ldots 2)^T$.

(b) $G_{2m} = G_{2m}(\mathcal{O}) = \{\mathcal{O}, \mathcal{E}(G_{2m})\}$, where

$$\mathcal{E}(G_{2m}) = \begin{bmatrix} \mathcal{E}(H_m) & \mathcal{E}(R_m) \\ \mathcal{E}(R_m) & \mathcal{E}(H_m) \end{bmatrix}.$$

A $(0,1)$-tiled order G_{2m} is Gorenstein with the permutation

$$\sigma = (1\, s+1)(2\, s+2)\ldots(s\, 2s).$$

(c) $\Gamma_{2m} = \Gamma_{2m}(\mathcal{O}) = \{\mathcal{O}, \mathcal{E}(\Gamma_m)\}$, where

$$\mathcal{E}(\Gamma_{2m}) = \begin{bmatrix} \mathcal{E}(H_m) & \mathcal{E}(R_m) \\ Y & \mathcal{E}(H_m) \end{bmatrix} \text{ and } Y = \begin{bmatrix} [P_2] \\ \vdots \\ [P_m] \\ [\pi P_2] \end{bmatrix}.$$

An order Γ_{2m} is a cyclic Gorenstein with the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & m & m+1 & \ldots & 2m-1 & 2m \\ m+1 & m+2 & \ldots & 2m & 2 & \ldots & m & 1 \end{pmatrix}.$$

(d) $D_m = D_m(\mathcal{O}) = \{\mathcal{O}, \mathcal{E}(D_m) = (\beta_{ij})\}$, where $\beta_{m1} = 2$ and all other elements β_{ij} coincide with elements α_{ij} of $\mathcal{E}(H_m)$ and

$$\Gamma_{2m+1} = \Gamma_{2m+1}(\mathcal{O}) = \{\mathcal{O}, \mathcal{E}(\Gamma_{2m+1})\},$$

where

$$\mathcal{E}(\Gamma_{2m+1}) = \left| \begin{array}{c|c} \mathcal{E}(D_{m+1}) & X \\ \hline Y & \mathcal{E}(H_m) \end{array} \right|$$

and

$$X = \begin{bmatrix} \mathcal{E}(R_m) \\ [\pi P_1] \end{bmatrix}, \quad Y = [[Q_m\pi]\, \mathcal{E}(R_m)].$$

An order Γ_{2m+1} is a cyclic Gorenstein with the permutation $\sigma =$

$$= \begin{pmatrix} 1 & 2 & \ldots & m & m+1 & m+2 & \ldots & 2m & 2m+1 \\ m+2 & m+3 & \ldots & 2m+1 & 1 & 2 & \ldots & m & m+1 \end{pmatrix}.$$

In particular case $m = 1$ we obtain

$$\mathcal{E}(\Gamma_3) = \begin{vmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \text{ and } \Gamma_3 \simeq \Delta_3, \text{ where } \mathcal{E}(\Delta_3) = \begin{vmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & 0 \end{vmatrix}.$$

Definition. Recall that a real $s \times s$–matrix $P = (p_{ij})$ is double stochastic if $\sum_{j=1}^{s} p_{ij} = \sum_{i=1}^{s} p_{ij} = 1$ for $i, j = 1, \ldots, s$.

It is easy to show that adjacency matrices of the quivers $Q(H_s)$, $Q(G_{2s})$, $Q(\Gamma_{2s})$ and $Q(\Gamma_{2s+1})$ have the form λP, where P is doubly stochastic and $\lambda = 1$ for $Q(H_s)$ and $\lambda = 2$ for other quivers.

Main Theorem. *A reduced Gorenstein tiled order has the hereditary ring of multipliers of the Jacobson radical if and only if it is isomorphic to one of the rings* $H_m(\mathcal{O})$, $G_{2m}(\mathcal{O})$, $\Gamma_{2m}(\mathcal{O})$ *or* $\Gamma_{2m+1}(\mathcal{O})$.

II. Gorenstein $(0, 1)$–orders.

In this section we shall use the notations and terminology of the paper [Sim2].

Definition. A tiled order $\Lambda = \{\mathcal{O}, \mathcal{E}(\mathcal{O}) = (\alpha_{ij})\}$ is called a $(0, 1)$–*order* if $\mathcal{E}(\Lambda)$ is a $(0, 1)$–matrix.

We associate with a reduced $(0, 1)$–order Λ the poset

$$I_\Lambda = \{1, \ldots, n\}$$

and the relation $\leq$ defined by the formula $i \leq j \Leftrightarrow \alpha_{ij} = 0$. It is easy to see that $(I_\Lambda, \leq)$ is a poset.
Conversely, with any finite poset

$$I = \{1, \ldots, n\}$$

we associate the reduced exponent $(0, 1)$–matrix $\mathcal{E}_I = (\gamma_{ij})$ by the following way: $\gamma_{ij} = 0 \Leftrightarrow$ if and only if $i \leq j$. Then $\Lambda(I) = \{\mathcal{O}, \mathcal{E}_I\}$ is a reduced $(0, 1)$–order.

Definition. The width of a poset I_Λ is called *the width of a reduced* $(0, 1)$–*order* Λ and is denoted $w(\Lambda)$.

In general case we define $w(\Lambda)$ as $w(\mathbf{M}(\Lambda))$ (see [ZK1 proposition 2.5]).

Theorem 2.1. *Any reduced Gorenstein* $(0,1)$*–order is isomorphic to a order* $H_m(\mathcal{O})$ *or to a order* $G_{2m}(\mathcal{O})$*.*

Proof. First of all we shall prove that the width $w(\Lambda)$ of Gorenstein $(0,1)$–order Λ is not greater 2.
Let $w(\Lambda) \geq 3$. Consequently there exist 3 pairwise non-comparable indecomposable modules P_i, P_j, P_k. Using the elementary transformation of type (2) let us assume $i = 1$, $j = 2$ and $k = 3$. Then

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 1 & \\ 1 & 0 & 1 & * \\ 1 & 1 & 0 & \\ & * & & * \end{pmatrix}.$$

Obviously, $\sigma(i) > 3$ for $i = 1,2,3$. As above, we can consider that $\sigma(1) = 4$, $\sigma(2) = 5$, $\sigma(3) = 6$. From the Gorenstein condition if follows that $\alpha_{i4} = 1-\alpha_{1i}, \alpha_{i5} = 1-\alpha_{2i}, \alpha_{i6} = 1-\alpha_{3i}$. First of all we shall compute the elements of $\mathcal{E}(\Lambda)$ for $i = 1,2,3$, after that — for $i = 4,5,6$. Therefore, the exponent matrix in this case is

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & * \\ 1 & 1 & 0 & 0 & 0 & 1 & \\ & & & 0 & 1 & 1 & \\ & * & & 1 & 0 & 1 & * \\ & & & 1 & 1 & 0 & \\ & * & & & * & & * \end{pmatrix}.$$

There are no symmetric zeroes in $\mathcal{E}(\Lambda)$. Since $\alpha_{42} = \alpha_{43} = \alpha_{51} = \alpha_{53} = \alpha_{61} = \alpha_{62} = 1$. From the unequalities $\alpha_{24}+\alpha_{41} \geq \alpha_{21}$, $\alpha_{15}+\alpha_{52} \geq \alpha_{12}$, $\alpha_{16}+\alpha_{63} \geq \alpha_{13}$ it follows $\alpha_{41} = \alpha_{52} = \alpha_{63} = 1$. Then

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & * \\ 1 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 0 & 1 & * \\ 1 & 1 & 1 & 1 & 1 & 0 & \\ & * & & & * & & * \end{pmatrix}.$$

Obviously that $\sigma(i) > 6$ for $i = 4,5,6$. We can consider $\sigma(4) = 7$, $\sigma(5) = 8$, $\sigma(6) = 9$. From the Gorenstein condition it follows $\alpha_{i7} = 1-\alpha_{4i}$, $\alpha_{i8} = 1-\alpha_{5i}$, $\alpha_{i9} = 1-\alpha_{6i}$. First of all we shall compute for $i = 1,2,3,4,5,6$, after that — for $i = 7, 8, 9$. Therefore, the exponent matrix in this case is

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & * \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & * \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\ & & & & & & 0 & 1 & 1 & \\ & * & & & * & & 1 & 0 & 1 & * \\ & & & & & & 1 & 1 & 0 & \\ & * & & & * & & & * & & * \end{pmatrix}.$$

Since are no symmetric zeroes in this matrix, then $\alpha_{71} = \alpha_{72} = \alpha_{73} = \alpha_{75} = \alpha_{76} = 1$, $\alpha_{81} = \alpha_{82} = \alpha_{83} = \alpha_{84} = \alpha_{86} = 1$, $\alpha_{91} = \alpha_{92} = \alpha_{93} = \alpha_{94} = \alpha_{95} = 1$. From the unequalities $\alpha_{57} + \alpha_{74} \geq \alpha_{54}$, $\alpha_{48} + \alpha_{85} \geq \alpha_{45}$, $\alpha_{49} + \alpha_{96} \geq \alpha_{46}$ it follows that $\alpha_{74} = \alpha_{85} = \alpha_{96} = 1$. Since,

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & * \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & * \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & * \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \\ & * & & & * & & & * & & * \end{pmatrix}.$$

Again, $\sigma(i) > 6$ for $i = 7, 8, 9$. Continuing this process we shall get that the exponent matrix $\mathcal{E}(\Lambda)$ have such a block form:

$$\mathcal{E}(\Lambda) = \begin{pmatrix} A & E & O & O & \dots & O & O \\ U & A & E & O & \dots & O & O \\ U & U & A & E & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ U & U & U & U & \dots & U & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \; U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix is not the exponent matrix of the reduced $(0,1)$–order because there does not exist such k that $\sigma(k) = 1,2,3$. Hence, the width of the poset $I_\Lambda \le 2$.

If $w(I_\Lambda) = 1$, in view of the theorem 3.4[ZK1] Λ is hereditary and then $\Lambda \simeq H_s(\mathcal{O})$.

Consider the case $w(I_\Lambda) = 2$, that means I_Λ has two non-comparable elements. Let they are P_1 and P_2. Then $\alpha_{12} = \alpha_{21} = 1$, and the exponent matrix has such a form:

$$\mathcal{E}(\Lambda) = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ * & * \end{pmatrix}.$$

Suppose, that $\sigma(1), \sigma(2) > 2$. One may assume, $\sigma(1) = 3$, $\sigma(2) = 4$. Then, in view of the Gorenstein condition, from $\alpha_{1j} + \alpha_{j3} = \alpha_{13} = 1$, $\alpha_{2j} + \alpha_{j4} = \alpha_{24} = 1$ obtain α_{j3} and α_{j4} for $j = 1,2,3,4$.

$$\mathcal{E}(\Lambda) = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & * \\ * & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ * & * & * \end{pmatrix}.$$

$\alpha_{32} = \alpha_{41} = 1$. From rings unequalities $\alpha_{14} + \alpha_{42} \ge \alpha_{12}$, $\alpha_{23} + \alpha_{31} \ge \alpha_{21}$ it follows, that $\alpha_{42} = \alpha_{31} = 1$. Then

$$\mathcal{E}(\Lambda) = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & * \\ \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & * \\ * & * & * \end{pmatrix}.$$

Obviously, $\sigma(3), \sigma(4) > 4$. Let $\sigma(3) = 5$, $\sigma(4) = 6$. from $\alpha_{i5} = 1 - \alpha_{3i}$ and $\alpha_{i6} = 1 - \alpha_{4i}$ obtain α_{i5} and α_{i6} for $i = 1, \dots, 6$. The matrix $\mathcal{E}(\Lambda)$ has the following form

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & \multirow{2}{*}{$*$} \\ 1 & 0 & 0 & 1 & 1 & 0 & \\ 1 & 1 & 0 & 1 & 1 & 0 & \multirow{2}{*}{$*$} \\ 1 & 1 & 1 & 0 & 0 & 1 & \\ \multicolumn{2}{c}{\multirow{2}{*}{$*$}} & \multicolumn{2}{c}{\multirow{2}{*}{$*$}} & 0 & 1 & \multirow{2}{*}{$*$} \\ & & & & 1 & 0 & \\ \multicolumn{2}{c}{*} & \multicolumn{2}{c}{*} & \multicolumn{2}{c}{*} & * \end{pmatrix}.$$

The exponent matrix does not have symmetric zeroes, then $\alpha_{51} = \alpha_{52} = \alpha_{54} = 1$, $\alpha_{61} = \alpha_{62} = \alpha_{63} = 1$. From $\alpha_{36} + \alpha_{64} \geq \alpha_{34}$, $\alpha_{45} + \alpha_{53} \geq \alpha_{43}$ it follows that $\alpha_{64} = \alpha_{53} = 1$.
Continuing this process we shall obtain that the exponent matrix has such block form:

$$\mathcal{E}(\Lambda) = \begin{pmatrix} A & E & O & O & \dots & O & O \\ U & A & E & O & \dots & O & O \\ U & U & A & E & \dots & O & O \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ U & U & U & U & \dots & U & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \; O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This matrix is not the exponent matrix of the reduced $(0,1)$–order because does not exist such k that $\sigma(k) = 1, 2, 3$.
Hence, at least, one of the numbers $\sigma(1)$, $\sigma(2)$ is less then 3. Suppose $\sigma(1) = 2$, but $\sigma(2) \neq 3$. Let $\sigma(2) = 3$. Then $\alpha_{i3} = 1 - \alpha_{2i}$ and

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 0 & \multirow{2}{*}{$*$} \\ 1 & 0 & 1 & \\ \multicolumn{2}{c}{*} & * & * \end{pmatrix}.$$

$\alpha_{ij} + \alpha_{ji} > 0$ and $\alpha_{13} + \alpha_{32} > \alpha_{12}$. Then, $\alpha_{31} = \alpha_{32} = 1$. Hence,

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 0 & \\ 1 & 0 & 1 & * \\ 1 & 1 & 0 & \\ & * & & * \end{pmatrix}.$$

Obviously, $\sigma(3) > 3$. We can consider $\sigma(3) = 4$. From the Gorenstein condition $\alpha_{3i} + \alpha_{4i} = 1$ obtain α_{i4} for $i = 1, 2, 3$. Then

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & * \\ 1 & 1 & 0 & 1 & \\ & & * & & * \end{pmatrix}.$$

$\alpha_{41} = \alpha_{42} = 1$ and $\alpha_{43} = 1$ because $\alpha_{24} + \alpha_{43} \geq \alpha_{23}$. Since,

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & \\ 1 & 1 & 0 & 1 & * \\ 1 & 1 & 1 & 0 & \\ & & * & & * \end{pmatrix}.$$

Continuing this process we obtain

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

This matrix is not the exponent matrix of reduced $(0,1)$–order because there does not exist such k that $\sigma(k) = 1$.

Hence, if P_i and P_j are two non-comparable elements of the poset I_Λ, then $\sigma(i) = j$ and $\sigma(j) = i$.

From this it follows that every element P_i may have only one non-comparable element, exactly it is $P_{\sigma(i)}$. Indeed, if P_i is non-comparable with P_j and P_k, then must be hold simultaneously $\sigma(i) = j$, $\sigma(j) = i$, $\sigma(i) = k$, $\sigma(k) = i$. It is possible only for $j = k$.

Let P_1 and P_{s+1}, P_2 and $P_{s+2}, \ldots, P_s$ and P_{2s} are s pairs of pairwise non-comparable elements of the poset I_Λ. $P_{2s+1}, \ldots, P_{2s+m}$ are the subset of elements of I_Λ, which are comparable with any element from I_Λ. The elements $P_{2s+1}, \ldots, P_{2s+m}$ are linearly ordered. The permutation σ decomposes into the product $\sigma = (1\, s+1)(2\, s+2)\ldots(s\, 2s)\sigma_m$, where the permutation σ_m acts on the set $\{2s+1, \ldots, 2s+m\}$.

Let $1 = e_1 + \ldots + e_{2s+m}$ be the decomposition of $1 \in \Lambda$ into the sum of pairwise orthogonal idempotents and $e_i\Lambda = P_i$. Denote $f_1 = e_1 + \ldots + e_{2s}$, $f_2 = 1 - f_1$. Since $P_{2s+1}, \ldots, P_{2s+m}$ is the subset of I_Λ and its width is equal 1, then $w(I_{f_2\Lambda f_2}) = 1$. Hence, $f_2\Lambda f_2 \simeq H_m(\mathcal{O})$. Therefore the exponent matrix has form:

$$\mathcal{E}(\Lambda) = \begin{pmatrix} \mathcal{E}(F) & * \\ * & \mathcal{E}(H) \end{pmatrix},$$

where $F = f_1 \Lambda f_1$. Since $\alpha_{2s+1\,\sigma(2s+1)} = 0$, then $\alpha_{2s+1\,j} = 0$ for any $j = 1, \dots, 2s+m$.

Lemma. *Let Λ be a reduced Gorenstein $(0,1)$–order with an exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})\, i,j = 1,\dots,s$ and a permutation σ, and let exists such i that $\alpha_{i\,\sigma(i)} = 0$. Then $\Lambda \simeq H_s(\mathcal{O})$, and $w(I_\Lambda) = 1$.*

Proof. Let $i = 1$, $\sigma(1) = s$. From $\alpha_{1j} + \alpha_{j\,\sigma(1)} = \alpha_{1\,\sigma(1)} = 0$ it follows that $\alpha_{1j} = 0$ for $j = 1,\dots,s$. It means P_1 is a smallest element in I_Λ, i.e. $P_1 \le P_j$ for any $j = 2,\dots,s$. An indecomposable projective right Λ-module P_1 has the unique maximal submodule $rad\,P_j$. Then, taking into account that $\mathcal{E}(\Lambda)$ is $(0,1)$-matrix, we have $P_1 \le rad\,P_1 \le P_j$ for $j = 2,\dots,s$. $P_1 = (0,\dots,0)$, $rad\,P_1 = (1,\,0,\,\dots,\,0)$. Then $\alpha_{1j} = 1$ for $j = 2,\dots,s$. The order Λ is Gorenstein, hence there exists such k, that $\sigma(k) = 1$. One can assume that $k = 2$. From $\alpha_{2j} + \alpha_{j1} = \alpha_{21} = 1$ we obtain $\alpha_{2j} = 0$ for $j = 2,\dots,s$. Then $P_1 \le rad\,P_1 = P_2 \le P_j$ for $j = 3,\dots,s$ and $P_2 \le rad\,P_2 \le P_j$ for $j = 2,\dots,s$. Since $rad\,P_2 = (1,\,1,\,0,\,\dots,\,0)$, then $\alpha_{j2} = 1$ for $j \ge 3$.
Let $\sigma(3) = 2$. Then $\alpha_{3j} + \alpha_{j2} = \alpha_{32} = 1$ and $\alpha_{3j} = 0$ for $j = 3,\dots,s$. Again, $rad\,P_s \le P_j$ for $j = 4,\dots,s$. Continuing this process we obtain such chain of the elements of I_Λ

$$P_1 \le rad\,P_1 = P_2 \le rad\,P_2 = P_3 \le \dots \le rad\,P_{s-1} = P_s,$$

The exponent matrix has the following form

$$\mathcal{E}(\Lambda) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

So $w(I_\Lambda) = 1$ and $\Lambda \simeq H_s(\mathcal{O})$. Lemma is proved.
Hence $w(I_\Lambda) = 2$. Then the existence of the zero row in the exponent matrix is the contradiction to lemma. Therefore, in I_Λ there are only s pairs pairwise non-comparable elements. Hence

$$I_(\Lambda) = \left\{ \begin{array}{ccccccccccc} P_1 & \to & P_2 & \to & P_3 & \to & \cdots & \to & P_{s-1} & \to & P_s \\ & \nearrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\searrow & & \nearrow\!\!\!\!\searrow & \\ P_{s+1} & \to & P_{s+2} & \to & P_{s+3} & \to & \cdots & \to & P_{2s-1} & \to & P_{2s} \end{array} \right\}.$$

Denote $e = e_1 + \ldots + e_s$, $f = 1 - e$. Subsets $\{P_1, \ldots, P_s\}$ and $\{P_{s+1}, \ldots, P_{2s}\}$ linearly ordered in I_Λ, then $I_{e\Lambda e}$ and $I_{f\Lambda f}$ are linearly ordered also. Hence $e\Lambda e \simeq f\Lambda f \simeq H_s(\mathcal{O})$. So

$$\mathcal{E}(\Lambda) = \begin{pmatrix} \mathcal{E}(H) & * \\ * & \mathcal{E}(H) \end{pmatrix}.$$

Proposition. *Let Λ — a reduced Gorenstein $(0,1)$–order with a exponent matrix $\mathcal{E}(\Lambda) = (\alpha_{ij})$, $i, j = 1, \ldots, s$ and a permutation σ. If for some i, j $\sigma(i) = j$, $\sigma(j) = i$ and $\alpha_{i\,\sigma(i)} = \alpha_{j\,\sigma(j)} = 1$, then $\alpha_{ik} = \alpha_{jk}$ for all $k \neq i, j$.*

Proof. A reduced order Λ does not have symmetric zeroes, hence $\alpha_{ik} + \alpha_{ki} \geq 1$ for $k \neq i$. From Gorenstein condition follows $\alpha_{ik} + \alpha_{k\sigma(i)} = 1$. We obtain $\alpha_{ki} \geq \alpha_{k\sigma(i)}$ for $k \neq i$. Analogously $\alpha_{kj} \geq \alpha_{k\sigma(j)}$ for $k \neq j$. So $\alpha_{ki} \geq \alpha_{kj}$ and $\alpha_{kj} \geq \alpha_{ki}$, that is $\alpha_{kj} = \alpha_{ki}$, if $k \neq i, j$. Then for the same i, j, k $\alpha_{ik} + \alpha_{ki} = \alpha_{ik} + \alpha_{kj} = \alpha_{ik} + \alpha_{k\sigma(i)} = 1$. Again $\alpha_{jk} + \alpha_{kj} = 1$. Hence $\alpha_{ik} = \alpha_{jk}$ for $k \neq i, j$. The proposition is proved.

The elements P_i and P_{s+i} are non-comparable. By proposition we obtain $\alpha_{ik} = \alpha_{s+ik}$ for any $k \neq i, s+i$. From this equality we get α_{ik} for $k > s$

$$\alpha_{ik} = \alpha_{s+ik} = \begin{cases} 0, & \text{if} s + i < k \\ 1, & \text{if } s + i > k \end{cases},$$

and α_{s+ik} for $k \leq s$

$$\alpha_{s+ik} = \alpha_{ik} = \begin{cases} 0, & \text{if} i < k \\ 1, & \text{if } i > k \end{cases}$$

It is clear that $\alpha_{is+i} = \alpha_{s+ii} = 1$. Since

$$\mathcal{E}(B) = \begin{pmatrix} 0 & 0 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 \\ 1 & 0 & 0 & \cdot & 0 & 1 & 1 & 0 & \cdot & 0 \\ 1 & 1 & 0 & \cdot & 0 & 1 & 1 & 1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & 0 & 1 & 1 & 1 & \cdot & 1 \\ 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 \\ 1 & 1 & 0 & \cdot & 0 & 1 & 0 & 0 & \cdot & 0 \\ 1 & 1 & 1 & \cdot & 0 & 1 & 1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & 0 \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & s & s+1 & \cdots & 2s \\ s+1 & s+2 & \cdots & 2s & 1 & \cdots & s \end{pmatrix}.$$

We enumerate the elements of I_Λ in such way that P_{2k-1} and P_{2k} be non-comparable for $k = 1, \dots, s$. Then the exponent matrix $\mathcal{E}(G_{2s})$ has the following form:

$$\mathcal{E}(G_{2s}) = \begin{pmatrix} A & O & O & \cdots & O & O \\ U & A & O & \cdots & O & O \\ U & U & A & \cdots & O & O \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ U & U & U & \cdots & A & O \\ U & U & U & \cdots & U & A \end{pmatrix},$$

where

$$U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},\ O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

III. General case

Let $\Lambda = \{\mathcal{O}, \mathcal{E}(\Lambda)\}$ be a tiled order with a Jacobson radical R.

Denote $\Lambda_r(R) = \{a \in M_n(T) | R_a \subseteq R\} (\Lambda_l(R) = \{b \in M_n(T) | bR \subseteq R\})$ the right (left) ring of multipliers of the Jacobson radical R.

Example. Let $\Lambda = H_m(\mathcal{O}), \sigma = (1\ m\ m-1\ \dots 3\ 2)$. Then $\Lambda(R) = \Lambda_r(R) = \Lambda_l(R) = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $j \neq \sigma(i)$ and $\beta_{i\sigma(i)} = \alpha_{i\sigma(i)} - 1$ $(i, j = 1, \dots, n)$, i.e. $\beta_{1m} = -1$, $\beta_{21} = \beta_{32} = \dots = \beta_{m\ m-1} = 0$. Hence $\Lambda(R)$ is a minimal hereditary order.

Main proposition. *Left and right rings of multipliers of the Jacobson radical R of a reduced Gorenstein tiled order Λ coincide.*

Proof. Let $\mathcal{E}(\Lambda) = (\alpha_{ij})$ be an exponent matrix of Λ and a be a permutation such that $\alpha_{ik} + \alpha_{k\sigma(i)} = \alpha_{i\sigma(i)}$ for $i, k = 1, \dots, n$. It is easy to verify that $\Lambda(R) = \Lambda_r(R) = \Lambda_l(R) = (\beta_{ij})$, where $\beta_{ij} = \alpha_{ij}$ for $j \neq \sigma(i)$ and $\beta_{i\sigma(i)} = \alpha_{i\sigma(i)} - 1 (i, j = 1, \dots, n)$.

Definition. A reduced Gorenstein tiled order Λ with the hereditary ring of multipliers of the Jacobson radical R will be called the GH - order.

Proposition 3.1. *Let Λ be GH - order. Then $w(\Lambda) \leq 2$.*

Proof. Denote $P_1, \dots, P_s$ all pairwise non-isomorphic indecomposable projective Λ - modules. Then the set of the modules $P_1, \dots, P_s$ and $P_1R, \dots, P_sR$ contains all non-isomorphic irreducible Λ- lattices. Hence $l(X/XR) \leq 2$ for any irreducible Λ - lattice X and by the Theorem 3.5 [ZK1] it follows that $w(\Lambda) \leq 2$.

If $w(\Lambda) = 1$ then by Theorem 3.4 we have that Λ is a hereditary order and $\Lambda \cong H_s(\mathcal{O})$.

Let $w(\Lambda) = 2$. We use theorem 3.6. [ZK1] which gives a description such tiled orders.

Taking into account that Λ is a GH -order we obtain that Λ is isomorphic to one of orders $G_{2m}(\mathcal{O}), \Gamma_{2m}(\mathcal{O})$ or $\Gamma_{2m+1}(\mathcal{O})$.

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