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ON POLAR CREMONA TRANSFORMATIONS

Alexandru Dimca

1. Introduction

In this note we discuss the topology of the gradient rational map $grad(f): \mathbb{P}^n \to \mathbb{P}^n$ associated with any non-zero homogeneous polynomial $f \in \mathbb{C}[x_0, ..., x_n]$. Such mappings were recently considered by Dolgachev, see [Do].

First we recall in section 2 the topological description of the degree of $grad(f)$ obtained by Stefan Papadima and the author in [DP], emphasizing in Theorem 3 the relation to the Bouquet Theorem of Lê in [Lê]. Proposition 5 shows that this degree is related not only to the topology of hypersurfaces but also to the topology of complete intersections.

In section 3 we give some key examples showing how one can go in both directions on the road connecting topology to algebra and get useful new insights each way.

In section 4 we discuss the evidence we have to support a challenging conjecture, and add a few remarks concerning the papers [Do] and [dPW].

2. The degree of the gradient

There is a gradient map associated to any non-constant homogeneous polynomial $f \in \mathbb{C}[x_0, ..., x_n]$ of degree d, namely

$$\phi_f = grad(f) : D(f) \to \mathbb{P}^n, \quad (x_0 : ... : x_n) \mapsto (f_0(x) : ... : f_n(x))$$

where $D(f) = \{x \in \mathbb{P}^n; f(x) \neq 0\}$ is the principal open set associated to f and $f_i = \frac{\partial f}{\partial x_i}$. This map corresponds to the polar Cremona transformations considered by Dolgachev in [Do].

Let $d(f) = deg(\phi_f)$ denote the degree of the gradient map. It is defined to be zero if the gradient map is not dominant, and for a dominant map ϕ_f one has the following equivalent definitions:

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(i) There is a Zariski open and dense subset U in $\mathbb{P}^n$ such that for all $u \in U$ the fiber $\phi_f^{-1}(u)$ has exactly $d(f)$ points;

(ii) the rational fraction field extension $\phi_f^* : K(\mathbb{P}^n) \to K(D(f))$ has degree $d(f)$, see Mumford [M], Proposition (3.17).

In particular, this implies that $d(f) = 1$ if and only if the gradient map ϕ_f induces a birational isomorphism of the projective space $\mathbb{P}^n$.

Note that in all the above we may replace the open set $D(f)$ by the larger open set $E(f) = \mathbb{P}^n \setminus \{x \in \mathbb{P}^n; f_0(x) = ... = f_n(x) = 0\}$ without changing the degree of the gradient map (to see this just use the description (i) given above for the degree). Note also that the gradient map is an affine morphism but not a finite morphism unless $V = V(f)$ is a smooth projective hypersurface and we work with the open set $E(f)$. Indeed, a finite morphism is proper and hence $E(f)$ has to be compact.

One of the main results in [DP] is the following topological description of the degree $d(f)$ of the gradient map $grad(f)$.

Theorem 1. *For any non-constant homogeneous polynomial $f \in \mathbb{C}[x_0, ..., x_n]$, the complement $D(f)$ is homotopy equivalent to a CW complex obtained from $D(f) \cap H$ by attaching $d(f)$ cells of dimension n, where H is a generic hyperplane in $\mathbb{P}^n$. In particular, one has*

$$d(f) = (-1)^n \chi(D(f) \setminus H).$$

Note that the meaning of 'generic' here is quite explicit: the hyperplane H

has to be transversal to a stratification of the projective hypersurface V. As remarked already in [DP] this Theorem gives a positive answer to Dolgachev's conjecture at the end of section 3 in [Do].

Corollary 2. *The degree of the gradient map $grad(f)$ depends only on the reduced polynomial f_{red} associated to f.*

It is a challeging question to find an algebraic proof for this result.

Moreover, Theorem 1. can be restated in the following way, which shows that for any projective hypersurface V, if we choose the hyperplane at infinity H in a generic way, then the topology of the affine part $X = V \setminus H$ is very simple.

Theorem 3. *For any non-constant homogeneous polynomial $f \in \mathbb{C}[x_0, ..., x_n]$, the affine part $X(f) = V(f) \setminus H$ of the corresponding projective hypersurface $V(f)$ with respect to a generic choice of the hyperplane at infinity H is homotopy equivalent to a bouquet of $(n-1)$-spheres. The number of spheres in this bouquet is the degree $d(f)$.*

Indeed, let $\ell = 0$ be a linear equation for the hyperplane H. Then the affine part $X(f)$ can be identified to the Milnor fiber of the germ $\ell : (\{x \in \mathbb{C}^{n+1}; f(x) = 0\}, 0) \to (\mathbb{C}, 0)$ using the usual passage from local to global objects in the presence of homogeneity, see for instance [D].

The linear form ℓ being generic, this germ has an isolated singularity at the origin in the stratified sense. The results in [Lê] imply that $X(f)$ is homotopy equivalent to a bouquet of $(n-1)$-spheres and our Theorem 1 shows that the number of spheres in this bouquet is precisely $d(f)$.

Remark 4. Using Thom's Second Isotopy Lemma, see for instance [D], it follows that for any projective variety V the topology of the affine part $X = V \setminus H$ is independent of H for a generic hyperplan H. For this reason we will use the alternative simpler notation V_a for the generic affine piece X of the projective variety V. Exactly the same argument as in the proof of Theorem 3 shows that V_a is homotopy equivalent to a bouquet of k-spheres when V is a complete intersection of dimension k. In the next result we explain how the number of spheres in such a bouquet can be computed using degrees of gradient maps in the case of a codimension two complete intersection. The general case is similar and is therefore left to the interested reader. For related results see Damon [Da].

Proposition 5. *Let $V = V(f, g)$ be a codimension two complete intersection in $\mathbb{P}^n$. Then*

$$b_{n-2}(V_a) = d(fg) - d(f) - d(g).$$

Proof. Note that $V(f) \cap V(g) = V$ and $V(f) \cup V(g) = V(fg)$ and use the general formula for the Euler numbers

$$\chi(A \cap B) + \chi(A \cup B) = \chi(A) + \chi(B)$$

in addition to Theorem 3.

3. Some interesting examples

Our first example is a polynomial f for which we know in advance that $d(f) = 1$ and use Theorem 3 to get information on the topology of the associated affine hypersurface V_a. Identify the space $\mathbb{C}^{n^2}$ to the space of square matrices of size n over $\mathbb{C}$ and let $f : \mathbb{C}^{n^2} \to \mathbb{C}$ be the determinant function, regarded as a homogeneous polynomial of degree n. Then it is known that $d(f) = 1$, see [Do], Example 3. In fact the corresponding gradient map in this case can be written in terms of matrices as $A \to A^{-1}$ and hence it is clearly a birational isomorphism!

The corresponding hypersurface $V(f)$ in $\mathbb{P}^{n^2}$ is very singular and has a rich topology due to the obvious fibration $\mathbb{C}^* \to Gl(n,\mathbb{C}) \to D(f)$. For instance, using the results in [DL] section 6. it follows that the weight polynomial of $D(f)$ is given by

$$W(D(f),t) = (1-t^4)(1-t^6)\cdot \ldots \cdot (1-t^{2n}).$$

This shows in particular that the sum of the Betti numbers of $D(f)$ (which is bounded below by the sum of the absolute values of the coefficients in the weight polynomial $W(D(f),t)$) or of $V(f)$ tends to infinity for n going to infinity.

Nonetheless the generic affine piece V_a is very simple, since Theorem 3. implies the following.

Corollary 6. *The generic affine piece $V(det)_a$ is homotopy equivalent to a (n^2-2)-sphere.*

Our second example deals with plane curves, i.e. homogeneous polynomials in $\mathbb{C}[x,y,z]$. By Corollary 2 we can and do assume that the polynomial f is reduced. Let $C = V(f)$ be the corresponding plane curve and let $C = C_1 \cup \ldots \cup C_m$ be the decomposition of the curve C into distinct irreducible components. The following result was obtained by Dolgachev using algebraic geometry methods in [Do].

Proposition 7. *The following two conditions are equivalent.*
(i) $d(f) = 1$;
(ii) one of the following holds:
(a) $m = 1$ and C is a smooth conic;
(b) $m = 2$ and C is a smooth conic plus a tangent;
(c) $m = 3$ and C is a union of three non-concurrent lines. Our

aim now is to give a topological proof for this beautiful result. Recall that an irreducible projective curve has the homotopy type of a smooth curve of genus g with a number k of attached circles S^1. In case this curve is a plane curve of degree d, it follows that the corresponding affine piece C_a satisfies $b_1(C_a) = 2g + k + d - 1$. Indeed C_a is a bouquet of circles and is obtained from C by deleting d smooth points.

Let's see which are the minimal values for this Betti number $b_1(C_a)$.

(A) $b_1(C_a) = 0$ iff $g = k = d - 1 = 0$, i.e. C is a line.

(B) $b_1(C_a) = 1$ iff $g = k = 0$ and $d = 2$, i.e. C is a smooth conic.

Now back to the proof of Proposition 7. The case C irreducible, i.e. $m = 1$ is clear by (B) above. Assume now that $m = 2$. Then the Mayer-Vietoris exact sequence in homology gives

$$b_1(C_{1a} \cup C_{2a}) = b_1(C_{1a}) + b_1(C_{2a}) + b_0(C_{1a} \cap C_{2a}) - 1$$

Using (A) and (B) we can find the minimal values for $b_1(C_a)$.

(C) $b_1(C_a) = 0$ iff both components C_i for $i = 1, 2$ are lines;

(D) $b_1(C_a) = 1$ iff one component is a smooth conic and the other is a line tangent to the conic.

This completes the proof in the case $m = 2$. A similar discussion shows that for $m = 3$ there is just one possibility (the triangle) giving $b_1(C_a) = 1$, while for $m > 3$ there are no such configurations.

4. Hypersurfaces with isolated singularities

The aim of this section is to give support for the following.

Conjecture 8. *Let $f \in \mathbb{C}[x_0, ..., x_n]$ be a reduced homogeneous polynomial such that*

(a) $d = deg(f) > 2$ and $n > 2$;

(b) the associated projective hypersurface $V(f)$ has only isolated singularities.

Then $d(f) \neq 1$.

Theorem 9. *This conjecture is true if all the singularities of the hypersurface $V(f)$ are in addition weighted homogeneous.*

Proof. Using Theorem 3. and known facts on the topology of special fibers in a deformation of an isolated hypersurface singularity we have

$$d(f) = (d-1)^n - \mu(V(f))$$

where $\mu(V(f))$ is the sum of the Milnor numbers of all the singularities of $V(f)$, see [D], p. 161 for details.

When all these singularities are weighted homogeneous, then

$$\mu(V(f)) = \tau(V(f))$$

where $\tau(V(f))$ is the sum of the Tjurina numbers of all the singularities of $V(f)$. Note that the left hand side of the equality in Lemma 2. in [Do] is exactly $\tau(V(f))$, hence in our opinion that result is valid only when all the singularities of $V(f)$ are in addition weighted homogeneous.

The proof of Theorem 9. is based on recent results by du Plessis and Wall [dPW], some of them to be recalled below.

Case 1. $V(f)$ is a cone, i.e. after a linear coordinate change the polynomial f is independent of x_0. In this case the gradient map ϕ_f is not dominant, and hence $d(f) = 0$.

Case 2. $V(f)$ is not a cone, i.e. f_0 does not belong to the ideal $(f_1, ..., f_n)$. Assume moreover that the coordinates are chosen such that $f_1, ..., f_n, x_0$ is a regular sequence in $\mathbb{C}[x_0, ..., x_n]$. Let $\overline{r}$ be the minimal degree of a homogeneous polynomial g such that g is not divisible by x_0 and $gf_0 \in (f_1, ..., f_n, x_0)$. It is obvious that $0 \leq \overline{r} \leq d-1$.

If $\overline{r} > 0$, then the Theorems (4.4) and (5.1) in [dPW] imply that

$$d(f) \geq \overline{r}(d-1-\overline{r})(d-1)^{n-2}.$$

This gives the result for $\overline{r} < d-1$. When $\overline{r} = d-1$, we can apply the final remark at the end of section 4 in [dPW] and get the inequality

$$d(f) \geq 2\binom{n+d-2}{n} \geq 2(n+1) \geq 8.$$

To complete the proof we show now that for a generic choice of the hyperplane $H : x_0 = 0$ one has $\overline{r} > 0$ for any hypersurface $V(f)$ with isolated singularities when $n > 2$.

Note that $\overline{r} = 0$ if and only if there is a linear combination $f_a = a_0 f_0 + ... + a_n f_n$ with $a_j \in \mathbb{C}$, $a_0 \neq 0$, such that the hypersurface $V(f_a)$ has the hyperplane H as an irreducible component.

Consider the linear system

$$S = \{f_b = b_0 f_0 + ... + b_n f_n; b \in \mathbb{P}^n\}$$

and note that this linear system does not depend (up-to automorphisms of the projective spaces with coordinates x and b) on the choice of the coordinates x. The base points of S are exactly the singular points $V(f)_{sing}$ of our hypersurface, hence a generic member of the linear system S is smooth outside $V(f)_{sing}$.

It follows that the set

$$S_0 = \{b \in \mathbb{P}^n; V(f_b) \textit{ contains a hyperplane}\}$$

is a closed proper subset of $\mathbb{P}^n$. In particular $dim(S_0) < n$. The algebraic subset

$$T_0 = \{(b, H); b \in S_0 \textit{ and } V(f_b) \textit{ contains the hyperplane } H\}$$

has $dim(T_0) = dim(S_0)$, since the first projection has finite fibers. It follows that $Y_0 = pr_2(T_0)$ is a closed proper subset of the dual projective space $\mathbb{P}^n$ parametrizing the hyperplanes. Hence a generic hyperplane is not a component for any hypersurface $V(f_b)$.

If we choose the coordinate system on $\mathbb{P}^n$ such that $x_0 = 0$ is an equation for a hyperplane not in Y_0, then we get $\overline{r} > 0$ as we have claimed.

Remark 10. The Conjecture 8. is also true when $V(f)$ is a $\mathbb{Q}$-homology manifold. Indeed, in this case it follows as above that

$$d(f) = (d-1)^n - \mu(V(f)) \geq (d-1)^n - b^0_{n-1}(W^d_{n-1})$$

where W^d_m denotes a smooth projective hypersurface of dimension m and degree d and b^0_j stands for the primitive Betti numbers, see Theorem (5.4.3) in [D] for details. Using the equality (5.3.27) in [D], we get

$$d(f) \geq b^0_{n-2}(W^d_{n-2}) > 1.$$

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Laboratoire de Mathématiques Pures de Bordeaux,
Université Bordeaux I,
33405 Talence Cedex,
France
e-mail: dimca@math.u-bordeaux.fr