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# THE DUAL OF THE CATEGORY OF GENERALIZED TREES

Şerban Basarab

**Abstract**

A set $T$ together with a symmetric ternary operation $Y : T^3 \to T$ is said to be a *median set* or a *generalized tree* if $Y(x, x, y) = x$ and $Y((Y(x, u, v), Y(y, u, v), z) = Y(Y(x, y, z), u, v)$ for all $x, y, z, u \in T$.

Extending suitably Stone's duality for distributive lattices it is shown that the category of median sets is dual to the category having as objects the systems $(X, 0, 1, \neg)$, where $X$ is an irreducible spectral space with generic point $0, 1$ is the unique closed point of $X$ and $\neg$ is a unary operation on $X$ satisfying the following conditions:

i) $\neg\neg x = x$ for all $x \in X$,

ii) for each quasi–compact open subset $U$ of $X$, the set

$$\neg U := \{x \in X : \neg x \notin U\}$$

is quasi–compact open too, and

iii) the quasi–compact open subsets $U$ of $X$ satisfying $\neg U = U$ generate the topology of $X$.

It is also shown that the category of median sets is equivalent to the category having as objects the systems $(A, \vee, \wedge, \neg)$, where $(A, \vee, \wedge)$ is a distributive lattice and $\neg$ is a unary operation on $A$ such that the following conditions are satisfied:

i) $\neg$ is a negation operator, i.e. $\neg\neg a = a$ and $\neg(a \vee b) = \neg a \wedge \neg b$ for $a, b \in A$, and

ii) the subset $T(A) = \{a \in A : \neg a = a\}$ of $A$ generates the lattice $A$.

## Introduction

As it is well known, in the last years various types of trees have been the subject of much investigation mixing intuitive geometric ideas with more sofisticated algebraic and geometric structures and methods.

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As J. Morgan remarks in [11], *"part of the drama of the subject is guessing what type of techniques will be appropriate for a given investigation : Will it be direct and simple notions related to schematic drawings of trees or will it be notions from the deepest parts of algebraic group theory, ergodic theory, or commutative algebra which must be brought to bear? Part of the beauty of the subject is that the naïve tree considerations have an impact on these more sophisticated topics. In addition, trees form a bridge between these disparate subjects".*

The author's interest in this topic began about ten years ago after reading by chance the fundamental paper [12], where a natural generalization of the simplicial trees (i.e. acyclic connected graphs) was introduced under the name of $\Lambda$–*trees*. This notion is obtained from that of simplicial tree, interpreted in a natural way as metric space with an integer–valued distance function, by replacing the ordered group $\mathbb{Z}$ by any totally ordered abelian group $\Lambda$.

In their study of group actions on $\Lambda$–trees [1], Alperin and Bass regarded the fundamental problem of the subject to be *"to find the group–theoretic information carried by a $\Lambda$–tree action, analogous to that presented in ( Serre's book)* **"Trees"** *for the case $\Lambda = \mathbb{Z}$."*

In the works [2, 3, 5] concerning the problem above, the author developed a technique having two complementary aspects : a group–theoretic one concerning group actions on groupoids and a metric one concerning length functions on groupoids with values in a lattice ordered abelian group $\Lambda$, with the goal to find suitable generalizations for the notions of trees, graphs, graphs of groups, universal covering relative to a graph of groups, etc.

Thus, based on the fact that in a simplicial tree for any three vertices $x, y, z$ there exist a unique vertex $Y(x, y, z)$ lying on the geodesics connecting any two of them, a general notion of tree is defined as a set $T$ together with a ternary operation $Y$ subject to three equational axioms as below, and a systematic study of these generalised trees is initiated and continued later in [6]. To his surprise, the author learned quite recently that this general notion of tree has been known for a long time under the name of *median algebra* and extensively studied, mainly in the context of universal algebra (see [7], [9], [14]). Moreover it seems that median algebras are little known amongst group theorists and only recently attracted their interest. Let us mention only the work [13], where the median algebras are used for an extended study of Dunwoody's construction and Sageev's theorem, based on the remark that Sageev's geometric characterization of cubings has an algebraic counter part.

The present paper is a slightly improved version of the preprint [4] devoted to the investigation of the dual category of the category of generalized trees. Let's mention that the main results of the preprint above were frequently used by the author in [6] and in more recent papers concerning the arboreal structures on groups and fields. Note also that, although using a different language, similar results were independently obtained by Roller in [13].

In the following we understand by a *generalized tree* or a *median set* a set $T$ together with a ternary operation $Y : T^3 \to T$ satisfying the following equational axioms:

**Symmetry:** $Y(x, y, z) = Y(y, x, z) = Y(x, z, y)$

**Absorptive law:** $Y(x, x, y) = x$

**Selfdistributive law:** $Y(Y(x, u, v), Y(y, u, v), z) = Y(Y(x, y, z), u, v)$.

It is shown in [5] that the $\Lambda$–trees as defined in [12], [1], [2] (in particular, the simplicial trees), the distributive lattices and Tits' buildings are natural examples of median sets.

The median sets form a category $MED$ having as morphisms the maps $f : T \to T'$ satisfying $f(Y(x, y, z)) = Y(f(x), f(y), f(z))$ for $x, y, z \in T$.

The main goal of the present paper is to describe the dual of the category $MED$. This task is achieved by extending suitably Stone's duality for distributive lattices.

## 1. Stone's duality for distributive lattices

By a *lattice* we understand a poset $A$ in which every *non-empty* finite subset $F$ of $A$ has both a join ( a least upper bound) $\vee F$ and a meet (a greatest lower bound) $\wedge F$. This is equivalent to saying that $A$ is equipped with two binary operations $\vee$ and $\wedge$ such that $(A, \vee)$ and $(A, \wedge)$ are semilattices (i.e. commutative semigroups in which every element is idempotent) satisfying the absorptive laws $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

Usually (as for instance in [8], [10]) lattices are assumed to have a least and a last element. However, from technical reasons, we are forced to consider in the following the general case.

The lattices form a category *Lat* having as morphisms the maps $f : A \to B$ satisfying $f(a \vee b) = f(a) \vee f(b)$, $f(a \wedge b) = f(a) \wedge f(b)$ for $a, b \in A$.

The lattice $A$ is said to be *distributive* if the *distributive law* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ holds for all $a, b, c \in A$. Note that in a distributive lattice the dual of the identity above is satisfied too.

Denote by *DLat* the full subcategory of *Lat* having as objects the distributive lattices. The empty lattice is an initial object of *DLat*, while the one-element lattice is a final object.

*Definition.* A subset $I$ of a lattice $A$ is called an *ideal* of $A$ if $a \in I$, $b \in I$ imply $a \vee b \in I$, and $I$ is a lower set, i.e. $a \in I$ and $b \leq a$ imply $b \in I$. A subset

$F$ of $A$ satisfying axioms dual to those defining an ideal is called a *filter*.

An ideal $I$ of the lattice $A$ is said to be *prime* if its complement in $A$ is a filter, i.e. $a \wedge b \in I$ implies either $a \in I$ or $b \in I$. The complement of a prime ideal is called a *prime filter*.

*Definition.* A topological space $X$ is said to be *spectral* (or *coherent*) if

i) $X$ is *sober*, i.e. every irreducible non-empty closed subset of $X$ is the closure of a unique point of $X$, and

ii) the family of all quasi–compact open subsets of $X$ is closed under finite intersection (in particular, $X$ itself is quasi-compact) and forms a base for the topology of $X$.

Denote by *IrrSpec* the category of the systems $(X, 0, 1)$, where $X$ is an irreducible spectral space with the generic point 0, having a unique closed point 1. The morphisms in *IrrSpec*, called *coherent maps*, are those continuous functions $f : X \to Y$ for which $f(0) = 0$, $f(1) = 1$, and $f^{-1}(U)$ is quasi-compact whenever $U$ is a quasi-compact open subset of $Y$.

**Theorem 1.1** (Stone's representation theorem for distributive lattices) *The category IrrSpec is dual to the category DLat.*

The duality sends an object $(X, 0, 1)$ of *IrrSpec* to the lattice of proper quasi-compact open subset $U$ of $X$ (*proper* means $U \neq \emptyset$ and $U \neq X$, equivalently, $0 \in U$ and $1 \notin U$), and a distributive lattice $A$ to the system $(Spec\ A, \emptyset, A)$, where the space *Spec A* is the *prime spectrum* of $A$. The points of *Spec A* are the prime ideals of $A$, while its open sets may be identified with arbitrary ideals of $A$, a point $P$ being in an open set $I$ iff $I \not\subseteq P$. The correspondence $a \mapsto U(a) = \{P \in SpecA : a \notin P\}$ establishes a lattice isomorphism of $A$ onto the lattice of proper quasi–compact open subsets of *Spec A*.

There is an alternative description of the dual of the category *DLat*.

*Definition.* By a *quasi–boolean lattice* we understand a distributive lattice $A$ in which for arbitrary $a, b, c \in A$ such that $a \leq c \leq b$ there exists (of course unique) $d \in A$ satisfying $c \wedge d = a$ and $c \vee d = b$, i.e. for all $a, b \in A$ such that $a \leq b$, the interval $[a, b] := \{c \in A : a \leq c \leq b\}$ is a boolean algebra.

Thus the boolean algebras are those quasi–boolean lattices which have both a least and a last element.

*Definition.* By a *quasi–boolean space* we understand an object $(X, 0, 1)$ of *IrrSpec* such that the subspace $X - \{0, 1\}$ satisfies the $T_1$-axiom, i.e. for all $x, y \in X$, $x \to y$ (i.e. $y$ is contained in the closure of $\{x\}$) implies either $x = 0$ or $y = 1$ or $x = y$.

The duality between *DLat* and *IrrSpec* induces by restriction a duality between the category of quasi–boolean lattices and the category of quasi–boolean spaces. In particular, the duals of boolean algebras are those objects $(X, 0, 1)$ for which $X - \{0, 1\}$ is a Stone space.

*Definition.* By an *ordered quasi–boolean space* we understand a quasi–boolean space $(X, 0, 1)$ together with a partial order $\leq$ such that for all $x, y \in X$ there exists a lower quasi–compact open subset $U$ of $X$ satisfying $x \notin U$ and $y \in U$, whenever $x \not\leq y$. It follows that $0 \leq x \leq 1$ for all $x \in X$.

The ordered quasi–boolean spaces with order preserving coherent maps form a category *OQBooleSp*.

**Theorem 1.2** *The categories IrrSpec and OQBooleSp are canonically isomorphic.*

*Proof* Given an object $(X, 0, 1)$ of *IrrSpec*, let $A$ be the lattice of quasi–compact open proper subsets $U$ of $X$, and let $B = \{U \cup \cup_{i=1}^{n}(V_i - W_i) : n \in \mathbb{N}; U, V_i, W_i \in A\}$. $B$ is a quasi–boolean lattice generated by its sublattice $A$. Moreover $B \cup \{X\}$ is a base of the so called *patch topology* on $X$, with respect to which $(X, 0, 1)$ becomes a quasi–boolean space whose quasi–compact open proper subsets are exactly the members of $B$. Considering the partial order on $X$ given by the specialization relation $\rightarrow$ with respect to the $A$–topology on $X$, we get the ordered quasi–boolean space associated to the object $(X, 0, 1)$ of *IrrSpec*. Note that the $A$–open sets of $X$ are identified with the lower (with respect to $\rightarrow$) $B$–open sets of $X$.

Conversely, given an ordered quasi–boolean space $(X, 0, 1, \leq)$, the lower open subsets of $X$ form a topology on $X$ with respect to which $(X, 0, 1)$ becomes an object of *IrrSpec*, while the specialization relation is identified with the partial order $\leq$. □

**Corollary 1.3** *The forgetful functor from the category of quasi–boolean lattices into DLat has a left adjoint assigning to any distributive lattice A the quasi-boolean lattice freely generated by A.*

## 2. Distributive lattices and irreducible spectral spaces with negation

*Definition.* By a *negation* on a distributive lattice $A$ we understand a unary operation $\neg : A \rightarrow A$ satisfying the following equational axioms:

*Double negation law:* $\neg\neg a = a$

*De Morgan law:* $\neg(a \vee b) = \neg a \wedge \neg b$.

Note that the identity $\neg(a \wedge b) = \neg a \vee \neg b$ holds too.

The distributive lattices with negation form a category *NDLat* having as morphisms the lattice morphisms $f : A \rightarrow B$ satisfying $f(\neg a) = \neg f(a)$ for $a \in$

$A$. The category of boolean algebras is identified with a non–full subcategory of *NDLat*.

*Definition.* By an *irreducible spectral space with negation* we understand an object $(X, 0, 1)$ of *IrrSpec* together with a map $\neg : X \to X$ subject to the following conditions:

i) $\neg\neg x = x$ for all $x \in X$, and

ii) for each quasi–compact open subset $U$ of $X$, the set $\neg U := \{x \in X : \neg x \notin U\}$ is quasi–compact open too.

The irreducible spectral spaces with negation form a category *NIrrSpec* having as morphisms the coherent maps $f : X \to Y$ satisfying $f(\neg x) = \neg f(x)$ for $x \in X$.

The next lemma is immediate.

**Lemma 2.1** *Let $(X, 0, 1, \neg)$ be an object of NIrrSpec, and let $x, y \in X$. Then $x \to y$ implies $\neg y \to \neg x$. In particular, $\neg 0 = 1$.*

**Lemma 2.2** *Let $A$ be a distributive lattice and $X = Spec\ A$ be its prime spectrum. There exists a canonical bijection between the negations on $A$ and the negations on $X$.*

*Proof.* Assume $\neg$ is a negation on $A$. Given a prime ideal $P$ of $A$, the subset $\neg P := \{a \in A : \neg a \notin P\}$ is a prime ideal too, and $\neg\neg P = P$. Let $D$ be a quasi–compact open subset of $X$. Then

$$\neg D = \begin{cases} X & \text{if} \quad D = \emptyset \\ \emptyset & \text{if} \quad D = X \\ U(\neg a) & \text{if} \quad D = U(a) \ \text{ for some } \ a \in A, \end{cases}$$

so $\neg D$ is quasi–compact open too. Thus we get a negation on $X$.

Conversely, given a negation $\neg$ on $X$, define the unary operation $\neg : A \to A$ by assigning to each $a \in A$ the unique element $\neg a \in A$ for which $\neg U(a) = U(\neg a)$. □

The next theorem is an immediate consequence of Theorem 1.1. and Lemma 2.2.

**Theorem 2.3** *The category NIrrSpec is the dual of the category NDLat. This duality induces by restriction a duality between the category of quasi–boolean lattices with negation and the category of quasi–boolean spaces with negation.*

To get an alternative description of the category *NIrrSpec* we need the following concept:

*Definition* By an *ordered quasi–boolean space with negation* we understand an ordered quasi–boolean space $(X, 0, 1, \leq)$ together with a negation $\neg : X \to X$ on the underlying quasi–boolean space $(X, 0, 1)$ which is compatible with the partial order $\leq$, i.e. $x \leq y$ implies $\neg y \leq \neg x$ for all $x, y \in X$.

The ordered quasi–boolean spaces with negation, with the order preserving coherent maps commuting with negation as morphisms, form a category *NOQBooleSp*.

As a consequence of Theorem 1.2. we get

**Theorem 2.4** *The categories NIrrSpec and NOQBooleSp are canonically isomorphic.*

**Theorem 2.5** *The forgetful functor from the category of quasi–boolean lattices with negation into NDLat has a left adjoint assigning to any distributive lattice with negation $(A, \neg)$ the quasi–boolean lattice with negation freely generated by $(A, \neg)$.*

Some particularly interesting full subcategories of *NDLat* and *NIrrSpec* are defined as follows.

*Definition* By a *quasi–linear lattice* we understand a distributive lattice with negation $(A, \neg)$ such that for each $a \in A$ either $a \leq \neg a$ or $\neg a \leq a$.

Denote by *QLinLat* the category of quasi–linear lattices.

*Definition.* An irreducible spectral space with negation $(X, 0, 1, \neg)$ is said to be *quasi–linear* if for all $x, y \in X$ either $x \to y$ or $y \to x$ or $x \to \neg y$ or $\neg x \to y$.

Denote by *QLinSpec* the category of quasi–linear irreducible spectral spaces.

**Proposition 2.6** *The duality $NDLat \to NIrrSpec$ induces by restriction a duality $QLinLat \to QLinSpec$.*

*Proof* Let $(A, \neg)$ be a distributive lattice with negation. We have to show that the necessary and sufficient condition for $(A, \neg)$ to be quasi–linear is that its dual $(Spec\ A, \emptyset, A, \neg)$ is quasi–linear.

First assume that $(A, \neg)$ is quasi–linear, and let $P, Q \in Spec\ A$ be such that $P \not\subseteq Q$, $Q \not\subseteq P$ and $P \not\subseteq \neg Q$. Let $d \in \neg P$, i.e. $\neg d \notin P$. We have to show that $d \in Q$. By hypothesis there exists $a \in P - Q$, $b \in Q - P$ and $c \in P$ such that $\neg c \in Q$. Let $e := (a \wedge d) \vee (\neg b \wedge c)$. As $(A, \neg)$ is quasi–linear, we distinguish two cases:

**Case 1:** $e \leq \neg e$. Then $a \wedge d \leq b \vee \neg c \in Q$, whence $a \wedge d \in Q$. Since by assumption $a \notin Q$ it follows that $d \in Q$.

**Case 2:** $\neg e \leq e$. Then $b \wedge \neg d \leq a \vee c \in P$, whence $b \wedge \neg d \in P$, contrary to the assumption that $b \notin P$ and $\neg d \notin P$.

Consequently, $e \leq \neg e$ and hence $d \in Q$ as contended.

Next assume that $(Spec\ A, \emptyset, A, \neg)$ is quasi–linear, and let $a \in A$ be such that $a \not\leq \neg a$. To conclude that $\neg a \leq a$ we have to show that for each $Q \in Spec\ A$, $a \in Q$ implies $\neg a \in Q$. Let $Q \in Spec\ A$ be such that $a \in Q$, so $\neg a \notin \neg Q$. By hypothesis there exists $P \in Spec\ A$ such that $\neg a \in P$ and $a \notin P$, so $\neg a \in P \cap \neg P$. As a $a \in Q - P$ and $\neg a \in P - \neg Q$ it follows that either $P \subseteq Q$ or $\neg P \subseteq Q$, whence $\neg a \in P \cap \neg P \subseteq Q$. □

## 3. Some basic properties of median sets

Let $T$ be a median set with the ternary operation $Y$.

*Definition.* A subset $I$ of $T$ is said to be an *ideal* (or a *convex* subset) of $T$ if for all $a, b, c \in T$, $a \in I$ and $b \in I$ imply $Y(a, b, c) \in I$.

In particular, any convex subset of a median set $T$ is a median subset.

As the intersection of a family of convex subsets of $T$ is also convex, we may speak on the *convex closure* of a subset $S$ of $T$ and denote it by $[S]$. Note that $[\emptyset] = \emptyset$, $[\{a\}] = \{a\}$ for $a \in T$, and $[\{a, b\}] := [a, b] = \{Y(a, b, c) : c \in T\} = \{c \in T : Y(a, b, c) = c\}$ for $a, b \in T$, cf [5] Lemma 2.5.

*Definition.* By a *cell* (or a *simplex*) of the median set $T$ we understand a convex subset $I$ of $T$ of the form $I = [a, b]$ with $a, b \in T$. Given a cell $I$, any $a \in T$ for which there exists $b \in T$ such that $I = [a, b]$ is called an *end* of the cell $I$. The (non-empty) subset of all ends of the cell $I$, denoted by $\partial I$, is called the *boundary* of the cell $I$.

According to [5] Lemma 2.5., the boundary $\partial I$ of a cell $I$ is a median subset of $I$ and there exists a canonical map $\partial I \to \partial I$, $a \mapsto \bar{a}$ such that $I = [a, \bar{a}]$, $\bar{\bar{a}} = a$ for $a \in \partial I$ and $\overline{Y(a, b, c)} = Y(\bar{a}, \bar{b}, \bar{c})$ for $a, b, c \in \partial I$. Given $a \in \partial I$, the cell $I$ becomes a distributive lattice with respect to the partial order $b \underset{a}{\subset} c$ iff $b \in [a, c]$, with the least element $a$ and the last element $\bar{a}$. The boundary $\partial I$ is identified with the boolean subalgebra of the distributive lattice $(I, \underset{a}{\subset})$ consisting of those elements which have (unique) complements.

Some useful elementary facts proved in [5] §2 are collected in the next proposition.

**Proposition 3.1** *Let $T$ be a median set.*

*a) $[a, b] \cap [a, c] = [a, Y(a, b, c)]$ and $[a, b] \cap [b, c] \cap [c, a] = \{Y(a, b, c)\}$ for $a, b, c \in T$.*

*b) $c \in [a, b]$ iff $[a, c] \cap [b, c] = \{c\}$.*

*c) Given $a \in T$, $T$ becomes a meet-semilattice with respect to the partial order $\underset{a}{\subset}$ given by $b \underset{a}{\subset} c$ if $b \in [a, c]$, with the meet $b \underset{a}{\cap} c = Y(a, b, c)$ and the least element $a$.*

*d) For* $a, b_1, \ldots, b_n \in T$, $n \geq 1$, $\cap_{i=1}^{n}[a, b_i] = [a, b]$, *where* $b = \underset{a}{\cap}\{b_1, \ldots, b_n\}$ *is the meet of the family* $\{b_1, \ldots, b_n\}$ *with respect to the order* $\underset{a}{\subset}$.

*e) For* $a_1, \ldots a_n \in T$, $n \geq 1$ *the convex closure* $[a_1, \ldots, a_n]$ *of the finite subset* $\{a_1, \ldots, a_n\}$ *equals* $\{\underset{a}{\cap}\{a_1, \ldots, a_n\} : a \in T\} = \{b \in T : \underset{b}{\cap}\{a_1, \ldots, a_n\} = b\}$

*f) Let* $a_1, \ldots a_n$, $b_1, \ldots, b_m \in T$ *be such that* $[a_1, \ldots, a_n] \cap [b_1, \ldots, b_m]$ *is nonempty. Then*

$$[a_1, \ldots, a_n] \cap [b_1, \ldots, b_m] = [\underset{a_1}{\cap}\{b_1, \ldots, b_m\}, \ldots, \underset{a_n}{\cap}\{b_1, \ldots, b_m\}] =$$

$$= [\underset{b_1}{\cap}\{a_1, \ldots, a_n\}, \ldots, \underset{b_m}{\cap}\{a_1, \ldots, a_n\}].$$

*In particular, for* $a, b, c, d \in T$, *either* $[a, b] \cap [c, d]$ *is empty or*

$$[a, b] \cap [c, d] = [Y(a, b, c), Y(a, b, d)] = [Y(a, c, d), Y(b, c, d)].$$

*g) For each subset* $S$ *of* $T$, *the convex subset* $[S]$ *is the union* $\cup_F [F]$, *where* $F$ *ranges over the family of all finite subset of* $S$.

*Definition.* A median set $T$ is said to be *locally boolean* if for every cell $I$ of $T$, $\partial I = I$, respectively *locally linear* if every cell of $T$ has at most two ends.

Let $T$ be a non–empty median set and let $a \in T$. Then, according to [5] Lemma 2.6., $T$ is locally boolean iff $T$ is a quasi–boolean lattice with respect to the order $\underset{a}{\subset}$.

Note also that a median set $T$ is locally linear iff the following equivalent conditions are satisfied :

i) For all $a, b, c \in T$, $c \in [a, b]$ implies $[a, b] = [a, c] \cup [c, b]$.

ii) For all $a, b \in T$, the partial order $\underset{a}{\subset}$ induces on the cell $[a, b]$ a total order with the least element $a$ and the last element $b$.

*Definition.* A median set $T$ is called *simplicial* or *discrete* if every cell of $T$ contains finitely many elements.

The simplicial trees, i.e. acyclic connected graphs, are identified with the locally linear simplicial median sets, and the subtrees of a simplicial tree $T$ correspond bijectively to the convex subsets of the median set canonically associated to $T$ .

## 4. From distributive lattices with negation to median sets

The category *DLat* of distributive lattices is naturally identified with a nonfull subcategory of the category *MED* of median sets. Given a distributive lattice $A$, the ternary operation $Y : A^3 \to A$, $(a, b, c) \mapsto Y(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a) = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$ is a median operation on $A$. Obviously, any lattice morphism is a morphism in the category *MED*.

If $\neg$ is a negation on a distributive lattice $A$ then this one is an automorphism of the underlying median set of $A$. Thus the category *NDLat* of distributive lattice with negation is identified with a non–full subcategory of the category *IMED* of *median sets with involution*; the objects of *IMED* are pairs $(T, s)$ consisting of a median set $T$ and an automorphism $s$ of $T$ subject to $s^2 = id_T$, while the morphisms $(T, s) \to (T', s')$ are morphisms $f : T \to T'$ in *MED* satisfying the equality $f \circ s = s' \circ f$.

By composing the forgetful functor $NDLat \to IMED$ with the functor $IMED \to MED$, $(T, s) \mapsto T^s = \{x \in T : sx = x\}$, we get a functor $\mathcal{T} : NDLat \to MED$, assigning to a distributive lattice with negation $(A, \neg)$ the median subset of $A$ with universe $\{a \in A : \neg a = a\}$.

**Lemma 4.1** *The functor* $\mathcal{T} : NDLat \to MED$ *induces by restriction a functor from the category* $NQBooleLat$ *of quasi–boolean lattices with negation to the category* $BMED$ *of locally boolean median sets, respectively a functor from the category* $QLinLat$ *of quasi–linear lattices to the category* $LMED$ *of locally linear median sets.*

*Proof.* Let $(A, \neg)$ be a quasi–boolean lattice with negation, and let $a, b, c \in T := \mathcal{T}(A)$ be such that $c$ belongs to the cell $[a, b]$ of $T$. In particular, $a \wedge b \leq c \leq a \vee b$. By assumption there exists a unique $d \in A$ such that $c \wedge d = a \wedge b$ and $c \vee d = a \vee b$. Applying the negation, we get $c \vee \neg d = a \vee b$ and $c \wedge \neg d = a \wedge b$, whence $d = \neg d \in T$. Moreover, it follows easily that the cells $[a, b]$ and $[c, d]$ of the median set $T$ coincide, concluding that the median set $T$ is locally boolean.

Next let $(A, \neg)$ be a quasi–linear lattice, and $a, b, c, d \in T$ be such that $c$ and $d$ belong to the cell $[a, b]$ of $T$, whence $a \wedge b \leq c \leq a \vee b$ and $a \wedge b \leq d \leq a \vee b$. We have to show that $d \in [a, c] \cup [c, b]$. Set $e = (a \wedge c) \vee (b \wedge d)$. By hypothesis we distinguish two cases.

**Case 1:** $e \leq \neg e$. Then $a \wedge c \leq b \vee d$, whence $a \wedge c \leq a \wedge (b \vee d) = (a \wedge b) \vee (a \wedge d) = a \wedge d \leq d$.

Applying the negation we get also $d \leq a \vee c$, and hence $d \in [a, c]$.

**Case 2:** $\neg e \leq e$. It follows that $a \wedge d \leq b \vee c$. Proceeding as in the case 1, we get $d \in [b, c]$. □

### 5. From median sets to irreducible spectral spaces with negation

The aim of this section is to construct a contravariant functor $Spec : MED \to NIrrSpec$ from the category of median sets to the category of irreducible spectral spaces with negation as defined in §2.

### 5.1. Shadows in median sets

*Definition* Given two subsets $A$ and $B$ of a median set $T$, let $Sh_A(B)$ be the subset of $T$ consisting of those $x \in T$ for which there exists $a \in A$ such that the intersection $[a, x] \cap B$ is non–empty. Call $Sh_A(B)$ the *shadow of $B$ with respect to $A$.*

In particular, for $A = \{a\}$ and $B = \{b\}$, $Sh_a(b) := Sh_A(B) = \{x \in T : b \in [a, x]\} = \{x \in T : b \underset{a}{\subset} x\}$.

The basic properties of the sets $Sh_A(B)$ for $A, B \subseteq T$ are collected in the following lemma.

**Lemma 5.1.1** *Let $A$ and $B$ be subsets of a median set $T$.*
*a) If $A$ is non–empty then $B \subseteq Sh_A(B)$.*
*b) $Sh_A(B) = \underset{a\in A, b\in B}{\cup} Sh_a(b)$.*
*c) $Sh_a(Sh_b(c)) = Sh_{Y(a,b,c)}(c)$ for $a, b, c \in T$.*
*d) If $A$ is a convex subset of $T$ then $Sh_A(Sh_A(B)) = Sh_A(B)$.*
*e) If $A$ is a convex subset of $T$ then the necessary and sufficient condition for $A$ and $Sh_A(B)$ to be disjoint is that $A$ and $B$ are disjoint.*
*f) If $A$ and $B$ are convex subsets of $T$ then $Sh_A(B)$ is convex too.*

*Proof.* The statements a) and b) are immediate.

c) Let $x \in Sh_a(Sh_b(c))$. Then there exists $y \in [a, x]$ such that $c \in [b, y]$. It follows that

$$Y(x, Y(a, b, c), c) = Y(x, Y(a, b, c), Y(y, b, c)) =$$

$$= Y(Y(x, a, y), b, c) = Y(y, b, c) = c,$$

i.e. $c \in [x, Y(a, b, c)]$, whence $x \in Sh_{Y(a,b,c)}(c)$.

Conversely, assuming $x \in Sh_{Y(a,b,c)}(c)$, we get

$$c = Y(x, Y(a, b, c), c) = Y(Y(x, a, c), Y(x, c, c), b) = Y(Y(x, a, c), b, c).$$

Setting $y = Y(x, a, c)$, it follows that $y \in [a, x]$ and $c \in [b, y]$, whence $x \in Sh_a(Sh_b(c))$.

Obviously, d) is a consequence of c).

e) Assume $A$ is convex and let $a \in A \cap Sh_A(B)$, i.e. $b \in [a', a]$ for some $a' \in A$, $b \in B$. Thus $b \in [a', a] \cap B \subseteq A \cap B$, whence $A \cap B$ is non–empty.

f) Assuming that $A$ and $B$ are convex, let $x, y \in Sh_A(B)$ and $z \in [x, y]$. Thus $b_1 \in [a_1, x]$ and $b_2 \in [a_2, y]$ for some $a_1, a_2 \in A$, $b_1, b_2 \in B$. To conclude that $z \in Sh_A(B)$ it suffices to show that $Y(b_1, b_2, z) \in [Y(a_1, a_2, Y(b_1, b_2, z)), z]$

since $Y(b_1, b_2, z) \in [b_1, b_2] \subseteq B$ and $Y(a_1, a_2, Y(b_1, b_2, z)) \in [a_1, a_2] \subseteq A$. Taking the point $z$ as a root of the median set $T$ and using the notation $\subset, \cap$ and $\cup$ instead of $\underset{z}{\subset}, \underset{z}{\cap}, \underset{z}{\cup}$, we get $x \cap y = Y(x, y, z) = z$, $Y(b_1, b_2, z) = b_1 \cap b_2 = Y(a_1, b_1, x) \cap Y(a_2, b_2, y) = (a_1 \cap a_2 \cap b_1 \cap b_2) \cup (a_1 \cap y \cap b_1 \cap b_2) \cup (a_1 \cap a_2 \cap b_1 \cap y) \cup (x \cap a_2 \cap b_1 \cap b_2) \cup (a_1 \cap x \cap a_2 \cap b_2) \subset (a_1 \cap a_2) \cup (a_1 \cap b_1 \cap b_2) \cup (a_2 \cap b_1 \cap b_2) = Y(a_1, a_2, b_1 \cap b_2) = Y(a_1, a_2, Y(b_1, b_2, z))$, as required. □

### 5.2. The fundamental existence theorem for prime ideals in median sets.

An ideal (i.e. a convex subset) $P$ of a median set $T$ is said to be *prime* if its complement in $T$ is also an ideal. Thus the complement $\neg P := T - P$ of a prime ideal $P$ of $T$ is a prime ideal too. In particular, the empty set $\emptyset$ and the whole $T$ are prime ideals.

Denote by $SpecT$ the non–empty set of all prime ideals of the median set $T$. Given a subset $A$ of $T$, set $V(A) = \{P \in SpecT : A \subseteq P\}$ and $U(A) = \{P \in SpecT : P \cap A = \emptyset\} = \{\neg P : P \in V(A)\}$. Obviously, $V(A) = V([A])$ and $U(A) = U([A])$ for each $A \subseteq T$.

**Theorem 5.2.1.** *Let $A$ and $B$ be subsets of a median set $T$. The necessary and sufficient condition for $V(A) \cap U(B)$ to be non–empty is that the intersection $[A] \cap [B]$ is empty.*

*Proof.* Assuming that $V(A) \cap U(B)$ is non–empty, let $P \in V(A) \cap U(B)$. Then $[A] \subseteq P$ and $[B] \subseteq \neg P$, whence $[A] \cap [B]$ is empty.

Conversely, assume $[A] \cap [B]$ is empty. We may assume that $A$ is non–empty since otherwise $\emptyset \in V(A) \cap U(B)$. By Zorn's lemma there exists an ideal $P$ of $T$ which is maximal amongst those containing the ideal $[A]$ and disjoint from the ideal $[B]$. According to Lemma 5.1.1. - the statements e) and f), $P = Sh_{[B]}(P)$. It remains to show that the ideal $P$ is prime. Let $x, y \in \neg P$ and $z \in [x, y]$. We have to show that $z \in \neg P$. Let $Q = [P \cup \{x\}]$. As $P$ is an ideal, it follows by Proposition 3.1. - the statements e) and g) - that $Q = \bigcup_{p \in P} [x, p]$. By the maximality of $P$ there exists $b \in [B] \cap Q$, i.e. $b \in [B] \cap [x, p]$ for some $p \in P$. On the other hand, since $z \in [x, y] \cap [p, z]$ and $b \in [x, p] \cap [b, y]$ it follows by Proposition 3.1. - the statement f) - that $Y(x, y, p) \in [p, z] \cap [b, y]$, whence $[p, z] \cap [b, y]$ is non–empty. Assuming $z \in P$, we get $[p, z] \subseteq P$ and hence $[b, y] \cap P$ is non–empty. Consequently, $y \in Sh_{[B]}(P) = P$, contrary to our assumption. Thus $z$ belongs to $\neg P$, as contended. □

**Corollary 5.2.2** For every subset $A$ of the median set $T$, $[A] = \bigcap_{P \in V(A)} P$.

### 5.3. The prime spectrum of a median set

Let $T$ be a median set and $X = \mathrm{Spec}\, T$ be the set of all prime ideals of $T$. The family of the subsets $U(A) = \{P \in X : P \cap A = \emptyset\}$ for $A$ ranging over the finite subsets of $T$ contains $X = U(\emptyset)$ and is closed under finite intersection, and hence is the base of a topology on $X$; call it the *spectral topology* on $X$. By Theorem 5.2.1. the map $I \mapsto U(I)$ induces a bijection of the set of convex closures of all finite subsets of $T$ onto the base above.

Note that the subfamily of basic open sets $U(a) = \{P \in X : a \notin P\}$ for $a \in T$ generates the spectral topology on $X$.

For each $P \in X$, the closure of $\{P\}$ is $V(P)$, i.e. the specialization relation on $X$ coincides with the inclusion of prime ideals. In particular, $X = V(\emptyset)$, i.e. $X$ is irreducible with the unique generic point $\emptyset$. On the other hand, as $\{T\} = V(T)$, $T$ is the unique closed point of $X$. Note also that $X$ is quasi–compact since $U(\emptyset) = X$ is the unique basic open set containing $T$.

**Lemma 5.3.1** *The space $X = \mathit{Spec}\, T$ is sober.*

*Proof.* We have to show that the function $P \mapsto V(P)$ maps bijectively $X$ onto the set of all non–empty irreducible closed subsets of $X$. The injectivity is obvious.

Assuming that $C$ is a non–empty irreducible closed subset of $X$, let $P = \bigcap_{Q \in C} Q$. We have to show that the ideal $P$ is prime and $P \in C$. Let $a, b \in \neg P$ and assume that there exists some $c \in [a, b] \cap P$. It follows that $C \subseteq V(c) \subseteq V(a) \cup V(b)$, whence, by the irreducibility of $C$, either $C \subseteq V(a)$ or $C \subseteq V(b)$, contrary to the assumption $a, b \in \neg P$. Therefore the ideal $P$ is prime.

It remains to show that $P \in C$. Assuming $P \notin C$, there exist $a_1, \ldots, a_n \in T$, $n \geq 1$, such that $P \in \cap_{i=1}^n U(a_i)$ and $C \subseteq \cup_{i=1}^n V(a_i)$. Since $C$ is irreducible, it follows that $C \subseteq V(a_{i_0})$ for some $i_0 \in \{1, \ldots, n\}$, whence $a_{i_0} \in P$, a contradiction. □

**Proposition 5.3.2** *The necessary and sufficient condition for an open subset $D$ of $X = \mathit{Spec}\ T$ to be quasi–compact is that $D$ is a finite union of basic open subsets of $X$.*

*Proof.* It suffices to show that for each finite non–empty subset $A$ of $T$, the basic open set $U(A)$ is quasi–compact. Assume $U(A) = \cap_{i=1} D_i$, where the $D_i$'s are open. Without loss we may assume that for each $i \in I$, $D_i = U(B_i)$ for some finite non–empty subset $B_i$ of $T$. Suppose that for each finite subset $F$ of $I$, $U(A) \nsubseteq \cup_{i \in F} D_i$. Let $M_F$ be the set of those functions $f : F \to \cup_{i \in F} B_i$ satisfying $f(i) \in B_i$ for $i \in F$ and $U(A) \nsubseteq \cup_{i \in F} U(f(i))$. By hypothesis, the finite sets $M_F$ are non–empty. The sets $M_F$ together with the restriction maps $M_{F_2} \to M_{F_1}$ for $F_1 \subseteq F_2$ form a directed inverse system, and hence the inverse limit $M = \lim\limits_{\leftarrow} M_F$ is non–empty. Consequently, there exists a function $f : I \to \cup_{i\in I} B_i$ such that $f(i) \in B_i$ for $i \in I$ and for each finite subset $F$ of $I$, $U(A) \cap V(f(F))$ is non–empty. According to Theorem 5.2.1., $[A] \cap [f(F)]$ is empty for each finite subset $F$ of $I$. As $f(I) = \cup_F f(F)$, we get $[f(I)] = \cup_F [f(F)]$ by Proposition 3.1. - the statement g) -, whence $[A] \cap [f(I)] = \emptyset$. By Theorem 5.2.1. again it follows that $U(A) \not\subseteq \cup_{i\in I} U(f(i))$, contrary to the assumption that $U(A) \subseteq \cup_{i\in I} D_i$. □

**Lemma 5.3.3** *The necessary and sufficient condition for a quasi–compact open proper subset $D$ of $X = Spec\, T$ to satisfy the equality $D = \neg D := \{P \in X : \neg P \notin D\}$ is that $D = U(a)$ for some (unique) $a \in T$.*

*Proof.* Obviously $\neg U(a) = U(a)$ for all $a \in T$.

Let $D$ be a quasi–compact open proper subset of $X$ and assume $D = \neg D$. By Proposition 5.3.2., $D$ has the form $\cup_{i=1}^n U(A_i)$ where $n \geq 1$ and the $A_i' s$ are non–empty finite subsets of $T$.

First let us show that $\cap_{i=1}^n [A_i]$ is non–empty. The case $n = 1$ is trivial so we may assume $n \geq 2$. Let $k \in \{1, \ldots, n\}$ be maximal with the property $\cap_{i=1}^k [A_i] = \emptyset$ and suppose $k < n$. By Theorem 5.2.1. there exists $P \in U(A_{k+1}) \cap V(\cap_{i=1}^k [A_i])$. Since $P \in U(A_{k+1}) \subseteq D = \neg D$, it follows that $\neg P \notin \cup_{i=1}^k U(A_i)$, i.e. $a_i \in \neg P$ for some $a_i \in A_i$, $1 \leq i \leq k$. Pick some $b$ in $\cap_{i=1}^k [A_i]$. Then $c := \bigcap\limits_b \{a_1, \ldots, a_k\} \subseteq \neg P$. On the other hand, $c \in \cap_{i=1}^k [b, a_i] \subseteq \cap_{i=1}^k [A_i] \subseteq P$, a contradiction.

Next let us show that $\cap_{i=1}^n [A_i]$ is a singleton. Assuming the contrary, let $a, b \in \cap_{i=1}^n [A_i]$ be such that $a \neq b$. By Theorem 5.2.1 there exists a prime ideal $P$ such that $a \in P$ and $b \in \neg P$. As $a \in P \cap \cap_{i=1}^n [A_i]$ it follows that $P \notin D$, and hence $\neg P \in \neg D = D$. Consequently, $b \notin \neg P$ since $b \in \cap_{i=1}^n [A_i]$, a contradiction.

Let $a$ be the unique element of the ideal $\cap_{i=1}^n [A_i]$. Obviously, $D \subseteq U(a)$, whence $U(a) = \neg U(a) \subseteq \neg D = D$, so $D = U(a)$ as contended. □

According to Lemma 5.3.1 and Proposition 5.3.2, $(Spec\ T, \emptyset, T, \neg)$ is an object of the category *NIrrSpec*. If $f : T \to T'$ is a morphism of median sets then the map $Spec\, T' \to Spec\, T$, $P' \mapsto f^{-1}(P')$ is a morphism in *NIrrSpec*, so we get a contravariant functor $Spec : MED \to NIrrSpec$. By Lemma 5.3.3, the spectral topology on *Spec* $T$ for any median set $T$ is generated by those quasi–compact open proper subsets $D$ of *Spec* $T$ satisfying $\neg D = D$.

**Lemma 5.3.4** *The functor Spec : $MED \to NIrrSpec$ induces by restriction a functor from the category BMED of locally boolean median sets (respectively from the category LMED of locally linear median sets) to the category NQBooleSp of quasi–boolean spaces with negation (respectively to the category QLinSpec of quasi–linear irreducible spectral spaces).*

*Proof.* First assume $T$ is a locally boolean median set and let $P, Q$ be proper prime ideals of $T$ such that $P \subseteq Q$. We have to show that $P = Q$. Assuming the contrary, there exist $a, b, c \in T$ such that $a \in P$, $c \in Q - P$ and $b \notin Q$. It follows that $Y(a, b, c) \in Q - P$ so we may assume from the beginning that $c \in [a, b]$. By hypothesis there exists $d \in T$ such that $[a, b] = [c, d]$. As $a \in P$, $c \notin P$ we get $d \in P \subseteq Q$, whence $b \in [c, d] \subseteq Q$, a contradiction.

Next assume $T$ is a locally linear median set and let $P, Q$ be prime ideals of $T$ such that $P \not\subseteq Q$, $Q \not\subseteq P$ and $P \not\subseteq \neg Q$. We have to show that $\neg P \subseteq Q$. By assumption there exist $a, b, c \in T$ such that $a \in P \cap \neg Q$, $b \in Q \cap \neg P$ and $c \in P \cap Q$. As $Y(a, b, c) \in [a, c] \cap [b, c] \subseteq P \cap Q$, we may assume $c \in [a, b]$. Let $d \in \neg P$. By hypothesis we distinguish two cases:

**Case 1:** $Y(a, b, d) \in [a, c]$. Then $Y(a, b, d) \in [a, c] \cap [b, d] \subseteq P \cap \neg P = \emptyset$, a contradiction.

**Case 2:** $Y(a, b, d) \in [c, b]$. Then $Y(a, b, d) \in Q \cap [a, d]$, whence $d \in Q$ since $a \notin Q$. □

## 6. The distributive lattice with negation freely generated by a median set.

By composing the contravariant functor $Spec : MED \to NIrrSpec$ as defined in §5 with the duality $NIrrSpec \to NDLat$ we get a covariant functor $\mathcal{L} : MED \to NDLat$ which assigns to a median set $T$ the distributive lattice of quasi-compact open proper subsets of $SpecT$ together with the negation $D \mapsto \neg D = \{P \in SpecT : \neg P \notin D\}$.

According to Theorem 2.3., Proposition 2.6. and Lemma 5.3.4., the functor $\mathcal{L}$ induces by restriction the functors $BMED \to NQBooleLat$, $LMED \to QLinLat$.

**Lemma 6.1** *There exists a canonical natural transformation $\eta : id_{MED} \to \mathcal{T} \circ \mathcal{L}$. Moreover $\eta$ is an isomorphism.*

*Proof.* Gicen a median set $T$, $\mathcal{T}(\mathcal{L}(T)) = \{U(a) : a \in T\}$ by Lemma 5.3.3. The canonical map $\eta(T) : T \to \mathcal{T}(\mathcal{L}(T))$, $a \mapsto U(a)$ is an isomorphism of median sets by Theorem 5.2.1. □

**Lemma 6.2** *There exists a canonical natural trasnsformation $\varepsilon : \mathcal{L} \circ \mathcal{T} \to id_{NDLat}$. Moreover $\varepsilon(A)$ is injective for any distributive lattice with negation $A$.*

*Proof.* Let $(A, \neg)$ be a distributive lattice with negation, and $T := \mathcal{T}(A, \neg)$ be the median subset of $A$ with universe $\{a \in A : \neg a = a\}$. The map $2^A \to 2^T$, $P \to P \cap T$ induces a morphism $Spec(A, \neg) \to Spec\, T$ in the category *NIrrSpec*.

Indeed, let $P$ be a prime ideal of $A$, and let $a, b, c \in T$. Assuming $a, b \in P$ it follows that $Y(a, b, c) \in P$ since $Y(a, b, c) \leq a \vee b \in P$. Assuming $a \notin P$, $b \notin P$ we get $Y(a, b, c) \notin P$ since $a \wedge b \leq Y(a, b, c)$ and $a \wedge b \notin P$. Consequently, $P \cap T$ is a prime ideal of the median set $T$. Note that $\neg P \cap T = T - P = \neg(P \cap T)$ for $P \in Spec\ A$; recall that $\neg P = \{a \in A : \neg a \notin P\}$ for $P \in Spec\ A$. As $\{P \in Spec\ A : P \cap T \in U_T(a)\} = U_A(a)$ for all $a \in T$, the map above is coherent.

By duality (Theorem 2.3), we get a morphism $\varepsilon(A) : \mathcal{L}(T) \to A$ of distributive lattices with negation. To conclude that $\varepsilon(A)$ is injective it suffices to show that the canonical map $Spec\ A \to Spec\ T$ is onto. Let $Q$ be a prime ideal of the median set $T$. Denote by $I$ the ideal of the lattice $A$ generated by $Q$, and by $F$ the filter of $A$ generated by $\neg Q = T - Q$.

**Claim:** The ideal $I$ is disjoint from the filter $F$.

Assuming the contrary, we get some $a_1, \ldots a_n \in \neg Q$, $b_1, \ldots, b_m \in Q$, $n \geq 1$, $m \geq 1$, such that $\wedge_{i=1}^n a_i \leq \vee_{j=1}^m b_j$. Set $a = \wedge_{i=1}^n a_i$, $b = \vee_{j=1}^m b_j$, $c = \bigcap_{b_1}\{a_1, \ldots, a_n\}$. We get $c = a \vee \vee_{i=1}^n (a_i \wedge b_1)$ and $\bigcap_{c}\{b_1, \ldots, b_m\} = (\wedge_{j=1}^m b_j) \vee \vee_{j=1}^m (b_j \wedge c) = (\wedge_{j=1}^m b_j) \vee (b \wedge c) = (\wedge_{j=1}^m b_j) \vee (a \wedge b) \vee \vee_{i=1}^n (a_i \wedge b_1) = c$, whence $c \in [a_1, \ldots, a_n] \cap [b_1, \ldots, b_m] \subseteq \neg Q \cap Q = \emptyset$, a contradiction.

Consequently, $I$ is disjoint from $F$ as claimed. According to the fundamental existence theorem for prime ideals in distributive lattices, there is a prime ideal $P$ of $A$ containing $I$ and disjoint from $F$, whence $P \cap T = Q$. □

**Theorem 6.3.** *The functor $\mathcal{L} : MED \to NDLat$ is a left adjoint of the functor $\mathcal{T} : NDLat \to MED$. In other words, $\mathcal{L}(T)$ is the distributive lattice with negation freely generated by a median set $T$.*

*Proof.* Since the natural transformations $\eta : id_{MED} \to \mathcal{T} \circ \mathcal{L}$ and $\varepsilon : \mathcal{L} \circ \mathcal{T} \to id_{NDLat}$ satisfy the triangular identities $\mathcal{T}(\varepsilon) \circ \eta(\mathcal{T}) = id_{\mathcal{T}}$ and $\varepsilon(\mathcal{L}) \circ \mathcal{L}(\eta) = id_{\mathcal{L}}$, $\mathcal{L}$ is a left adjoint of $\mathcal{T}$ having unit $\eta$ and counit $\varepsilon$. □

As a consequence of Theorem 6.3 and Lemmata 6.1 and 6.2 we get:

**Theorem 6.4** *a) The functor $\mathcal{L} : MED \to NDLat$ induces an equivalence between the category MED of median sets and the full subcategory of NDLat consisting of those distributive lattices with negation $(A, \neg)$ which are generated as lattices by their median subsets $\mathcal{T}(A) = \{a \in A : \neg a = a\}$.*

*b) The contravariant functor $Spec : MED \to NIrrSpec$ induces a duality between the category MED and the full subcategory of NIrrSpec consisting of those irreducible spectral spaces with negation $(X, 0, 1, \neg)$ whose topology is generated by the quasi–compact open subsets $D$ satisfying $\neg D = D$.*

As a consequence of the statement above it follows easily that any finitely generated median set is finite. In particular, any median set is a directed direct limit of finite median sets, and the convex closure of a finite subset of a simplicial median set is finite too.

**6.5. Examples**

i) Let $(A, \vee, \wedge)$ be a distributive lattice. The product $A \times A$ becomes a distributive lattice with negation with respect to the operations

$$(a,b) \vee (a',b') = (a \vee a', b \wedge b')$$

$$(a,b) \wedge (a',b') = (a \wedge a', b \vee b')$$

$$\neg(a,b) = (b,a).$$

The diagonal embedding $a \mapsto (a,a)$ identifies the underlying median set of $A$ with the median subset of the lattice above consisting of those elements which are invariant under the negation $\neg$. Thus we get the distributive lattice with negation freely generated by the underlying median set of $A$. Note also that the prime ideals of the underlying median set of $A$ correspond bijectively to the pairs $(P, F)$ consisting of a prime ideal $P$ and a prime filter $F$ of the distributive lattice $A$.

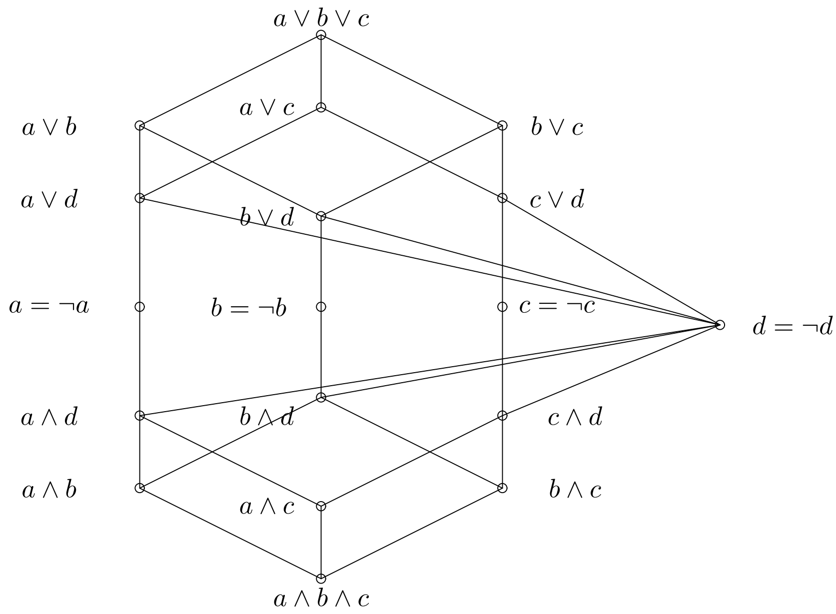
ii) Let $T$ be the tree with 4 distinct elements $a, b, c, d$ such that $Y(a,b,c) = d$. The prime spectrum of $T$ has 8 points, namely the prime ideals $\emptyset$, $\{a\}, \{b\}, \{c\}$ and their complements in $T$. The distributive lattice with negation $\mathcal{L}(T)$ freely generated by the tree $T$ has 18 distinct elements and is described by the diagram from the figure on the next page.

**7. The distributive lattice freely generated by a median set**

Let $t : DLat \to MED$ be the forgetful functor which identifies the category of distributive lattices with a non–full subcategory of the category of median sets. Denote by $l : MED \to DLat$ the functor obtained by composing the functor $\mathcal{L} : MED \to NDLat$ as defined in §6 with the forgetful functor $NDLat \to DLat$. The functor $l$ assigns to a median set $T$ the distributive lattice $l(T)$ of quasi–compact open proper subsets of the spectral space $Spec\, T$. Note that $T$ is identified with a median subset of $l(T)$ which generates $l(T)$ as a lattice.

**Proposition 7.1** *The functor $l : MED \to DLat$ is a left adjoint of the forgetful functor $t : DLat \to MED$. In other words, $l(T)$ is the distributive lattice freely generated by a median set $T$.*

*Proof.* Let $T$ and $A$ be a median set and respectively a distributive lattice. Given a morphism $f : T \to A$ of median sets, we have to extend it uniquely




to a morphism $\bar{f} : l(T) \to A$ of lattices.

As for each prime ideal $P$ of the distributive lattice $A$, $f^{-1}(P)$ is a prime ideal of the median set $T$, we get a map $f_* : \mathit{Spec}\ A \to \mathit{Spec}\ T$. One checks easily that $f_*$ is a morphism in the category *IrrSpec*. By Stone's duality, we get the required lattice morphism $\bar{f} : l(T) \to A$. □

## 8. The locally boolean median set freely generated by a median set

Given a median set $T$, let $(\mathit{Spec}\ T, \emptyset, T, \neg)$ be the irreducible spectral space with negation associated to $T$. Denote by $\mathcal{B}(T)$ the median subset of the power set $2^{\mathit{Spec}\ T}$ consisting of those proper subsets $D$ of $\mathit{Spec}\ T$ which are quasi–compact and open with respect to the topology on $\mathit{Spec}\ T$ (cf. §1) and satisfy the condition $\neg D = D$. The median set $T$ is identified with a median subset of $\mathcal{B}(T)$, and $\mathcal{B}(T)$ is locally boolean according to Lemma 4.1. Thus we get a functor $\mathcal{B} : MED \to BMED$.

**Proposition 8.1** *The functor $\mathcal{B}$ is a left adjoint of the forgetful functor $BMED \to MED$.*

*Proof.* Immediate by Theorems 2.4. and 6.4. Indeed, we get a duality between the category *MED* of median sets and the full subcategory of *NOQBooleSp* consisting of those ordered quasi–boolean spaces with negation $(X, 0, 1, \leq \neg)$ which satisfy the following condition: the lattice of lower quasi–compact open proper subsets of $X$ is generated by its members $D$ for which $\neg D = D$. □

**8.2 Remark** Let $(X, 0, 1, \neg)$ be the dual of a locally boolean median set, and $\leq$ be a partial order on $X$ making $(X, 0, 1, \leq, \neg)$ an ordered quasi–boolean space with negation. The lattice of lower quasi–compact open proper subsets of $X$ is not necessarily generated by its members $D$ satisfying $\neg D = D$. For instance, let $T$ be the tree consisting of two distinct points $x, y$. Then $X =$ *Spec* $T$ has four points, namely $\emptyset$, $T$, $\{x\}$, $\{y\}$, while the open proper subsets of $X$ are $U(x) = \{\emptyset, \{y\}\}$, $U(y) = \{\emptyset, \{x\}\}$, $U(x) \cap U(y) = \{\emptyset\}$ and $U(x) \cup U(y) = \{\emptyset, \{x\}, \{y\}\}$; they form a boolean lattice $\mathcal{L}(T)$ with 4 elements together with the negation $\neg$ given by $\neg U(x) = U(x)$, $\neg U(y) = U(y)$, $\neg(U(x) \cap U(y)) = U(x) \cup U(y)$. Consider the total order $\emptyset \leq \{x\} \leq \{y\} \leq T$ on $X$, making $(X, \emptyset, T, \leq, \neg)$ an object of the category *NOQBooleSp*. The lower open proper subsets of $X$ form the chain $U(x) \cap U(y) \subseteq U(x) \subseteq U(x) \cup U(y)$, whose unique member $D$ satisfying $\neg D = D$ is $U(x)$.

**8.3 Example** Let $T$ be the tree with four distinct elements $a, b, c, d$ such that $Y(a, b, c) = d$. The embedding $a \mapsto \{a\}$; $b \mapsto \{b\}$, $c \mapsto \{c\}$, $d \mapsto \emptyset$ identifies $T$ with a median subset of the power set of the set with three elements $\{a, b, c\}$ whose underlying median set is the locally boolean median set freely generated by $T$.

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Institute of Mathematics
of the Romanian Academy,
P.O.Box 1-764,
RO-70109 Bucharest,
Romania
e-mail: serban.basarab@imar.ro