DOI: 10.1515/auom-2016-0020
An. Şt. Univ. Ovidius Constanţa **Vol. 24(1),2016, 335–351**

ON 2-ABSORBING PRIMARY SUBMODULES OF MODULES OVER COMMUTATIVE RINGS

Hojjat Mostafanasab, Ece Yetkin, Ünsal Tekir and Ahmad Yousefian Darani*

Abstract

All rings are commutative with $1 \neq 0$, and all modules are unital. The purpose of this paper is to investigate the concept of 2-absorbing primary submodules generalizing 2-absorbing primary ideals of rings. Let M be an R-module. A proper submodule N of an R-module M is called a *2-absorbing primary submodule* of M if whenever $a, b \in R$ and $m \in M$ and $abm \in N$, then $am \in M\text{-rad}(N)$ or $bm \in M\text{-rad}(N)$ or $ab \in (N :_R M)$. It is shown that a proper submodule N of M is a 2-absorbing primary submodule if and only if whenever $I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M, then $I_1I_2 \subseteq (N :_R M)$ or $I_1K \subseteq M\text{-}rad(N)$ or $I_2K \subseteq M\text{-}rad(N)$. We prove that for a submodule N of an R-module M if $M\text{-}rad(N)$ is a prime submodule of M, then N is a 2-absorbing primary submodule of M. If N is a 2-absorbing primary submodule of a finitely generated multiplication R-module M, then $(N :_R M)$ is a 2-absorbing primary ideal of R and $M\text{-}rad(N)$ is a 2-absorbing submodule of M.

1 Introduction and Preliminaries

Throughout this paper all rings are commutative with a nonzero identity and all modules are considered to be unitary. Prime submodules have an important

Key Words: Multiplication module, Primary submodule, Prime submodule, 2-absorbing submodule, n-absorbing submodule.
2010 Mathematics Subject Classification: Primary 13A15; Secondary 13F05, 13G05.
Received: 08.09.2014.
Accepted: 20.10.2014.

role in the theory of modules over commutative rings. Let M be a module over a commutative ring R. A *prime* (resp. *primary*) submodule is a proper submodule N of M with the property that for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M)$ (resp. $a^k \in (N :_R M)$ for some positive integer k). In this case $p = (N :_R M)$ (resp. $p = \sqrt{(N :_R M)}$) is a prime ideal of R. There are several ways to generalize the concept of prime submodules. Weakly prime submodules were introduced by Ebrahimi Atani and Farzalipour in [16]. A proper submodule N of M is *weakly prime* if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. Behboodi and Koohi in [13] defined another class of submodules and called it weakly prime. Their paper is on the basis of some recent papers devoted to this new class of submodules. Let R be a ring and M an R-module. A proper submodule N of M is said to be *weakly prime* when for $a, b \in R$ and $m \in M$, $abm \in N$ implies that $am \in N$ or $bm \in N$. To avoid the ambiguity, Behboodi renamed this concept and called submodules introduced in [13], *classical prime submodule*.

Badawi in [9] generalized the concept of prime ideals in a different way. He defined a nonzero proper ideal I of R to be a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This definition can obviously be made for any ideal of R. This concept has a generalization, called weakly 2-absorbing ideals, which has studied in [10]. A proper ideal I of R to be a *weakly 2-absorbing ideal of* R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Anderson and Badawi [6] generalized the concept of 2-absorbing ideals to n-absorbing ideals. According to their definition, a proper ideal I of R is called an *n-absorbing* (resp. *strongly n-absorbing*) ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$ (resp. $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the x_i's (resp. n of the I_i's) whose product is in I. They proved that a proper ideal I of R is 2-absorbing if and only if I is strongly 2-absorbing.

In [26], the concept of 2-absorbing and weakly 2-absorbing ideals generalized to submodules of a module over a commutative ring. Let M be an R-module and N a proper submodule of M. N is said to be a *2-absorbing submodule* (resp. *weakly 2-absorbing submodule*) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. Badawi et. al. in [11] introduced the concept of 2-absorbing primary ideals, where a proper ideal I of R is called *2-absorbing primary* if whenever $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $ac \in \sqrt{I}$ or $bc \in \sqrt{I}$.

Let R be a ring, M an R-module and N a submodule of M. We will denote by $(N :_R M)$ *the residual of N by M*, that is, the set of all $r \in R$ such that $rM \subseteq N$. The annihilator of M which is denoted by $ann_R(M)$ is $(0 :_R M)$. An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$ [17]. Finitely generated faithful multiplication modules are cancellation modules [25, Corollary to Theorem 9], where an R-module M is defined to be a *cancellation module* if $IM = JM$ for ideals I and J of R implies $I = J$. It is well-known that if R is a commutative ring and M a nonzero multiplication R-module, then every proper submodule of M is contained in a maximal submodule of M and K is a maximal submodule of M if and only if there exists a maximal ideal $\mathfrak{m}$ of R such that $K = \mathfrak{m}M$ [17, Theorem 2.5]. If M is a finitely generated faithful multiplication R-module (hence cancellation), then it is easy to verify that $(IN :_R M) = I(N :_R M)$ for each submodule N of M and each ideal I of R. For a submodule N of M, if $N = IM$ for some ideal I of R, then we say that I is a presentation ideal of N. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [3, Theorem 3.4], the product of N and K is independent of presentations of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [3]). Let N be a proper submodule of a nonzero R-module M. Then the M-radical of N, denoted by M-rad(N), is defined to be the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then we say M-rad$(N) = M$. It is shown in [17, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad$(N) = \sqrt{(N :_R M)}M$. In this paper we define the concept of 2-absorbing primary submodules. We give some basic results of this class of submodules and discuss on the relations among 2-absorbing ideals, 2-absorbing submodules, 2-absorbing primary ideals and 2-absorbing primary submodules.

2 Properties of 2-absorbing primary submodules

Definition 2.1. A proper submodule N of an R-module M is called a *2-absorbing primary submodule* (resp. *weakly 2-absorbing primary submodule*) of M if whenever $a, b \in R$ and $m \in M$ and $abm \in N$ (resp. $0 \neq abm \in N$), then $am \in M$-rad(N) or $bm \in M$-rad(N) or $ab \in (N :_R M)$.

Example 2.2. Let p be a fixed prime integer and $N_0 = \mathbb{N} \cup \{0\}$. Each proper $\mathbb{Z}$-submodule of $\mathbb{Z}(p^\infty)$ is of the form $G_t = \langle 1/p^t + \mathbb{Z}\rangle$ for some $t \in N_0$. In [15, Example 1] it was shown that every submodule G_t is not primary. For each $t \in N_0$, $(G_t :_{\mathbb{Z}} \mathbb{Z}(p^\infty)) = 0$. Note that $p^2\left(\frac{1}{p^{t+2}} + \mathbb{Z}\right) = \frac{1}{p^t} + \mathbb{Z} \in G_t$, but neither $p^2 \in (G_t :_{\mathbb{Z}} \mathbb{Z}(p^\infty)) = 0$ nor $p\left(\frac{1}{p^{t+2}} + \mathbb{Z}\right) \in G_t$. Hence $\mathbb{Z}(p^\infty)$ has no 2-absorbing submodule. Since every prime submodule is 2-absorbing, then $\mathbb{Z}(p^\infty)$ has no prime submodule. Therefore $\mathbb{Z}(p^\infty)$-rad$(G_t) = \mathbb{Z}(p^\infty)$, and so G_t is a 2-absorbing primary submodule of $\mathbb{Z}(p^\infty)$.

Theorem 2.3. *Let N be a proper submodule of an R-module M. Then the following conditions are equivalent:*

1. *N is a 2-absorbing primary submodule of M;*
2. *For every elements $a, b \in R$ such that $ab \notin (N :_R M)$, $(N :_M ab) \subseteq (M$-$rad(N) :_M a) \cup (M$-$rad(N) :_M b)$;*
3. *For every elements $a, b \in R$ such that $ab \notin (N :_R M)$, $(N :_M ab) \subseteq (M$-$rad(N) :_M a)$ or $(N :_M ab) \subseteq (M$-$rad(N) :_M b)$.*

Proof. (1)⇒(2) Suppose that $a, b \in R$ such that $ab \notin (N :_R M)$. Let $m \in (N :_M ab)$. Then $abm \in N$, and so either $ma \in M$-$rad(N)$ or $bm \in M$-$rad(N)$. Therefore either $m \in (M$-$rad(N) :_M a)$ or $m \in (M$-$rad(N) :_M b)$. Hence $(N :_M ab) \subseteq (M$-$rad(N) :_M a) \cup (M$-$rad(N) :_M b)$.
(2)⇒(3) Notice to the fact that if a submodule (a subgroup) is a subset of the union of two submodules (two subgroups), then it is a subset of one of them. Thus we have $(N :_M ab) \subseteq (M$-$rad(N) :_M a)$ or $(N :_M ab) \subseteq (M$-$rad(N) :_M b)$.
(3)⇒(1) is straightforward. □

Lemma 2.4. *Let M be a finitely generated multiplication R-module. Then for any submodule N of M, $\sqrt{(N :_R M)} = (M$-$rad(N) :_R M)$.*

Proof. By [21, Theorem 4], $(M$-rad$(N) :_R M) \subseteq \sqrt{(N :_R M)}$. Now we prove the other containment without any assumption on M. Let K be a prime submodule of M containing N. Then clearly $(K : M)$ is a prime ideal that contains $(N : M)$. Therefore $\sqrt{(N :_R M)} \subseteq (K : M)$, so $\sqrt{(N :_R M)} \subseteq (M$-rad$(N) :_R M)$. □

Proposition 2.5. *Let M be a finitely generated multiplication R-module and N be a submodule of M. Then M-$rad(N)$ is a primary submodule of M if and only if M-$rad(N)$ is a prime submodule of M.*

Proof. Suppose that M-rad(N) is a primary submodule of M. Let $a \in R$ and $m \in M$ be such that $am \in M$-rad(N) and $m \notin M$-rad(N). Since M-rad(N) is primary, it follows $a \in \sqrt{(M\text{-rad}(N) :_R M)} = \sqrt{\sqrt{(N :_R M)}} = \sqrt{(N :_R M)} = (M$-rad$(N) :_R M)$, by Lemma 2.4. Thus M-rad(N) is a prime submodule of M. The converse part is clear. □

Theorem 2.6. *Let M be a finitely generated multiplication R-module. If N is a 2-absorbing primary submodule of M, then*

1. $(N :_R M)$ *is a 2-absorbing primary ideal of R.*
2. *M-rad(N) is a 2-absorbing submodule of M.*

Proof. (1) Let $a, b, c \in R$ be such that $abc \in (N :_R M)$, $ac \notin \sqrt{(N :_R M)}$ and $bc \notin \sqrt{(N :_R M)}$. Since, by Lemma 2.4, $\sqrt{(N :_R M)} = (M\text{-rad}(N) :_R M)$, there exist $m_1, m_2 \in M$ such that $acm_1 \notin M\text{-rad}(N)$ and $bcm_2 \notin M\text{-rad}(N)$. But $ab(cm_1 + cm_2) \in N$, because $abc \in (N :_R M)$. So $a(cm_1 + cm_2) \in M$-rad(N) or $b(cm_1 + cm_2) \in M\text{-rad}(N)$ or $ab \in (N :_R M)$, since N is 2-absorbing primary. If $ab \in (N :_R M)$, then we are done. Thus assume that $a(cm_1 + cm_2) \in M\text{-rad}(N)$. So $acm_2 \notin M\text{-rad}(N)$, because $acm_1 \notin M$-rad(N). Therefore $ab \in (N :_R M)$, since N is 2-absorbing primary and $abcm_2 \in N$. Similarly if $b(cm_1 + cm_2) \in M\text{-rad}(N)$, then $ab \in (N :_R M)$. Consequently $(N :_R M)$ is a 2-absorbing primary ideal.
(2) By [11, Theorem 2.3] we have two cases.

Case 1. $\sqrt{(N :_R M)} = p$ is a prime ideal of R. Since M is a multiplication module, $M\text{-rad}(N) = \sqrt{(N :_R M)}M = pM$, where pM is a prime submodule of M by [17, Corollary 2.11]. Hence in this case M-rad(N) is a 2-absorbing submodule of M.

Case 2. $\sqrt{(N :_R M)} = p_1 \cap p_2$, where p_1, p_2 are distinct prime ideals of R that are minimal over $(N :_R M)$. In this case, we have $M\text{-rad}(N) = \sqrt{(N :_R M)}M = (p_1 \cap p_2)M = ([p_1 + \text{ann}M] \cap [p_2 + \text{ann}M])M = p_1M \cap p_2M$, where p_1M, p_2M are prime submodules of M by [17, Corollary 2.11, 1.7]. Consequently, M-rad(N) is a 2-absorbing submodule of M by [26, Theorem 2.3]. □

Theorem 2.7. *Let M be a (resp. finitely generated multiplication) R-module and N be a submodule of M. If M-rad(N) is a (resp. primary) prime submodule of M, then N is a 2-absorbing primary submodule of M.*

Proof. Suppose that M-rad(N) is a prime submodule of M. Let $a, b \in R$ and $m \in M$ be such that $abm \in N$, $am \notin M\text{-rad}(N)$. Since M-rad(N) is a prime submodule and $abm \in M\text{-rad}(N)$, then $b \in (M\text{-rad}(N) :_R M)$. So $bm \in M$-rad(N). Consequently N is a 2-absorbing primary submodule of M. Now assume that M is a finitely generated multiplication module and M-rad(N) is a primary submodule of M, then M-rad(N) is a prime submodule of M, by Proposition 2.5. Therefore N is 2-absorbing primary. □

In [2, Theorem 1(3)], it was shown that for any faithful multiplication module M not necessary finitely generated, $M\text{-rad}(IM) = \sqrt{I}M$ for any ideal I of R.

Theorem 2.8. *Let M be a (resp. finitely generated faithful multiplication) faithful multiplication R-module. If M-rad(N) is a (resp. primary) prime submodule of M, then N^n is a 2-absorbing primary submodule of M for every positive integer $n \geq 1$.*

Proof. Assume that M is a (resp. finitely generated faithful multiplication) faithful multiplication module and M-rad(N) is a (resp. primary) prime submodule of M. There exists an ideal I of R such that $N = IM$. Thus

$$M - \text{rad}(N^n) = \sqrt{I^n}M = M - \text{rad}(N),$$

which is a (resp. primary) prime submodule of M. Hence for every positive integer $n \geq 1$, N^n is a 2-absorbing primary submodule of M, by Theorem 2.7. □

Recall that a commutative ring R with $1 \neq 0$ is called a divided ring if for every prime ideal p of R, we have $p \subseteq xR$ for every $x \in R \backslash p$. Generalizing this idea to modules we say that an R-module M is divided if for every prime submodule N of M, $N \subseteq Rm$ for all $m \in M \backslash N$.

Theorem 2.9. *If M is a divided R-module, then every proper submodule of M is a 2-absorbing primary submodule of M. In particular, every proper submodule of a chained module is a 2-absorbing primary submodule.*

Proof. Let N be a proper submodule of M. Since the prime submodules of a divided module are linearly ordered, we conclude that M-rad(N) is a prime submodule of M. Hence N is a 2-absorbing primary submodule of M by Theorem 2.7. □

Remark 2.10. Let $I = (0 :_R M)$ and $R' = R/I$. It is easy to see that N is a 2-absorbing primary R-submodule of M if and only if N is a 2-absorbing primary R'-submodule of M. Also, $(N :_R M)$ is a 2-absorbing primary ideal of R if and only if $(N :_{R'} M)$ is a 2-absorbing primary ideal of R'.

Theorem 2.11. *Let S be a multiplicatively closed subset of R and M be an R-module. If N is a 2-absorbing primary submodule of M and $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a 2-absorbing primary submodule of $S^{-1}M$.*

Proof. If $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s} \in S^{-1}N$, then $ua_1a_2m \in N$ for some $u \in S$. It follows that $ua_1m \in M$-rad(N) or $ua_2m \in M$-rad(N) or $a_1a_2 \in (N :_R M)$, so we conclude that $\frac{a_1}{s_1}\frac{m}{s} = \frac{ua_1m}{us_1s} \in S^{-1}(M\text{-rad}(N)) \subseteq S^{-1}M\text{-rad}(S^{-1}N)$ or $\frac{a_2}{s_2}\frac{m}{s} = \frac{ua_2m}{us_2s} \in S^{-1}M\text{-rad}(S^{-1}N)$ or $\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{a_1a_2}{s_1s_2} \in S^{-1}(N :_R M) \subseteq (S^{-1}N :_{S^{-1}R} S^{-1}M)$. □

Theorem 2.12. *Let I be a 2-absorbing primary ideal of a ring R and M a faithful multiplication R-module such that $Ass_R(M/\sqrt{I}M)$ is a totally ordered set. Then $abm \in IM$ implies that $am \in \sqrt{I}M$ or $bm \in \sqrt{I}M$ or $ab \in I$ whenever $a, b \in R$ and $m \in M$.*

Proof. Let $a, b \in R$, $m \in M$ and $abm \in IM$. If $(\sqrt{I}M :_R am) = R$ or $(\sqrt{I}M :_R bm) = R$, we are done. Suppose that $(\sqrt{I}M :_R am)$ and $(\sqrt{I}M :_R bm)$ are proper ideals of R. Since $\mathrm{Ass}_R(M/\sqrt{I}M)$ is a totally ordered set, $(\sqrt{I}M :_R am) \cup (\sqrt{I}M :_R bm)$ is an ideal of R, and so there is a maximal ideal $\mathfrak{m}$ such that $(\sqrt{I}M :_R am) \cup (\sqrt{I}M :_R bm) \subseteq \mathfrak{m}$. We have $am \notin \mathrm{T}_{\mathfrak{m}}(M) := \{m' \in M : (1-x)m' = 0$, for some $x \in \mathfrak{m}\}$, since $am \in \mathrm{T}_{\mathfrak{m}}(M)$ implies that $(1-x)am = 0$ for some $x \in \mathfrak{m}$, thus $(1-x)am \in \sqrt{I}M$ and so $1 - x \in (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$, a contradiction. So by [17, Theorem 1.2], there are $x \in \mathfrak{m}$ and $m' \in M$ such that $(1-x)M \subseteq Rm'$. Thus, $(1-x)m = rm'$ some $r \in R$. Moreover, $(1-x)abm = sm'$ for some $s \in I$, because $abm \in IM$. Hence $(abr - s)m' = 0$ and so $(1-x)(abr-s)M \subseteq (abr-s)Rm' = 0$. Thus $(1-x)(abr-s) = 0$, because M is faithful. Therefore, $(1-x)abr = (1-x)s \in I$. Then $(1-x)ar \in \sqrt{I}$ or $(1-x)b \in \sqrt{I}$ or $abr \in I$, since I is 2-absorbing primary. If $(1-x)ar \in \sqrt{I}$, then $(1-x)a \in \sqrt{I}$ or $(1-x)r \in \sqrt{I}$ or $ar \in \sqrt{I}$, because by [11, Theorem 2.2] $\sqrt{I}$ is a 2-absorbing ideal of R. If $(1-x)a \in \sqrt{I}$, then $(1-x)am \in \sqrt{I}M$ and so $1 - x \in (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$ that is a contradiction. If $(1-x)r \in \sqrt{I}$, then $(1-x)^2 m = (1-x)rm' \in \sqrt{I}M$ which implies that $(1-x)^2 \in (\sqrt{I}M :_R m) \subseteq (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$, a contradiction. Similarly we can see that $(1-x)b \notin \sqrt{I}$. Now, $ar \in \sqrt{I}$ implies that $(1-x)am = arm' \in \sqrt{I}M$ and so $1 - x \in (\sqrt{I}M :_R am) \subseteq \mathfrak{m}$ which is a contradiction.
If $arb \in I$, then $ar \in \sqrt{I}$ or $br \in \sqrt{I}$ or $ab \in I$ which the first two cases are impossible, thus $ab \in I$. □

Let R be a ring with the total quotient ring K. A nonzero ideal I of R is said to be *invertible* if $II^{-1} = R$, where $I^{-1} = \{x \in K \mid xI \subseteq R\}$. The concept of an invertible submodule was introduced in [23] as a generalization of the concept of an invertible ideal. Let M be an R-module and let $S = R\backslash\{0\}$. Then $T = \{t \in S \mid tm = 0$ for some $m \in M$ implies $m = 0\}$ is a multiplicatively closed subset of R. Let N be a submodule of M and $N' = \{x \in R_T \mid xN \subseteq M\}$. A submodule N is said to be *invertible in* M, if $N'N = M$, [23]. A nonzero R-module M is called *Dedekind* provided that each nonzero submodule of M is invertible.

We recall from [20] that, a finitely generated torsion-free multiplication module M over a domain R is a Dedekind module if and only if R is a Dedekind domain.

Theorem 2.13. *Let R be a Noetherian domain, M a torsion-free multiplication R-module. Then the following statements are equivalent:*

1. *M is a Dedekind module;*

2. *If N is a nonzero 2-absorbing primary submodule of M, then either $N = \mathfrak{M}^n$ for some maximal submodule $\mathfrak{M}$ of M and some positive integer $n \geq 1$ or $N = \mathfrak{M}_1^n \mathfrak{M}_2^m$ for some maximal submodules $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of M and some positive integers $n, m \geq 1$;*

3. *If N is a nonzero 2-absorbing primary submodule of M, then either $N = P^n$ for some prime submodule P of M and some positive integer $n \geq 1$ or $N = P_1^n P_2^m$ for some prime submodules P_1 and P_2 of M and some positive integers $n, m \geq 1$.*

Proof. By the fact that every multiplication module over a Noetherian ring is a Noetherian module, M is Noetherian and so finitely generated.
(1) $\Rightarrow$ (2) Let N be a 2-absorbing primary submodule of M. There exists a proper ideal I of R such that $N = IM$. So $(N :_R M) = I$ is a 2-absorbing primary ideal of R, by Theorem 2.6. Since R is a Dedekind domain, then we have either $I = \mathfrak{m}^n$ for some maximal ideal $\mathfrak{m}$ of R and some positive integer $n \geq 1$ or $I = \mathfrak{m}_1^n \mathfrak{m}_2^m$ for some maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of R and some positive integers $n, m \geq 1$, by [9, Theorem 2.11]. Thus, either $N = \mathfrak{m}^n M = (\mathfrak{m}M)^n$ or $N = (\mathfrak{m}_1 M)^n (\mathfrak{m}_2 M)^m$ as desired.
(2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1) It is sufficient to show that R is a Dedekind domain, for this let $\mathfrak{m}$ be a maximal ideal of R. Let I be an ideal of R such that $\mathfrak{m}^2 \subset I \subset \mathfrak{m}$. So $\sqrt{I} = \mathfrak{m}$ and then M-rad$(IM) = \mathfrak{m}M$, since M is a faithful multiplication R-module. Then IM is a 2-absorbing primary submodule of M, Theorem 2.7. By assumption, either $IM = P^n$ for some prime submodule P of M and some positive integer $n \geq 1$ or $IM = P_1^n P_2^m$ for some prime submodules P_1 and P_2 of M and some positive integers $n, m \geq 1$. Now, since M is cancellation, either $I = \mathfrak{p}^n$ for some prime ideal $\mathfrak{p}$ of R or $I = \mathfrak{p}_1^n \mathfrak{p}_2^m$ for some prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R, which any two cases have a contradiction. Hence there are no ideals properly between $\mathfrak{m}^2$ and $\mathfrak{m}$. Consequently R is a Dedekind domain by [19, Theorem 39.2, p. 470]. □

Proposition 2.14. *Let M be a multiplication R-module and K, N be submodules of M. Then*

1. $\sqrt{(KN :_R M)} = \sqrt{(K :_R M)} \cap \sqrt{(N :_R M)}$.
2. *M-rad$(KN) = M$-rad$(K) \cap M$-rad(N).*
3. *M-rad$(K \cap N) = M$-rad$(K) \cap M$-rad(N).*

Proof. (1) By hypothesis there exist ideals I, J of R such that $K = IM$ and $N = JM$. Now assume $r \in \sqrt{(K :_R M)} \cap \sqrt{(N :_R M)}$. Therefore there exist positive integers m, n such that $r^m M \subseteq IM$ and $r^n M \subseteq JM$. Hence $r^{m+n} M \subseteq r^m JM \subseteq IJM = KN$. So $r \in \sqrt{(KN :_R M)}$. Consequently $\sqrt{(K :_R M)} \cap \sqrt{(N :_R M)} \subseteq \sqrt{(KN :_R M)}$. The other inclusion trivially holds.
(2) By part (1) and [17, Corollary 1.7],

$$\begin{aligned} M - \mathrm{rad}(KN) &= \sqrt{(KN :_R M)}M = (\sqrt{(K :_R M)} \cap \sqrt{(N :_R M)})M \\ &= ([\sqrt{(K :_R M)} + \mathrm{ann}M] \cap [\sqrt{(N :_R M)} + \mathrm{ann}M])M \\ &= \sqrt{(K :_R M)}M \cap \sqrt{(N :_R M)}M \\ &= M - \mathrm{rad}(K) \cap M - \mathrm{rad}(N). \end{aligned}$$

(3) See [1, Theorem 15(3)]. □

Theorem 2.15. *Let M be a multiplication R-module and $N_1, N_2, \dots, N_n$ be 2-absorbing primary submodules of M with the same M-radical. Then $N = \cap_{i=1}^n N_i$ is a 2-absorbing primary submodule of M.*

Proof. Notice that M-rad$(N) = \cap_{i=1}^n M$-rad(N_i), by Proposition 2.14. Suppose that $abm \in N$ for some $a, b \in R$ and $m \in M$ and $ab \notin (N :_R M)$. Then $ab \notin (N_i :_R M)$ for some $1 \le i \le n$. Hence $am \in M$-rad(N_i) or $bm \in M$-rad(N_i). □

Lemma 2.16. *Let M be an R-module and N a 2-absorbing primary submodule of M. Suppose that $abK \subseteq N$ for some elements $a, b \in R$ and some submodule K of M. If $ab \notin (N :_R M)$, then $aK \subseteq M$-rad(N) or $bK \subseteq M$-rad(N).*

Proof. Suppose that $aK \nsubseteq M$-rad(N) and $bK \nsubseteq M$-rad(N). Then $ak_1 \notin M$-rad(N) and $bk_2 \notin M$-rad(N) for some $k_1, k_2 \in K$. Since $abk_1 \in N$ and $ab \notin (N :_R M)$ and $ak_1 \notin M$-rad(N), we have $bk_1 \in M$-rad(N). Since $abk_2 \in N$ and $ab \notin (N :_R M)$ and $bk_2 \notin M$-rad(N), we have $ak_2 \in M$-rad(N). Now, since $ab(k_1 + k_2) \in N$ and $ab \notin (N :_R M)$, we have $a(k_1 + k_2) \in M$-rad(N) or $b(k_1 + k_2) \in M$-rad(N). Suppose that $a(k_1 + k_2) = ak_1 + ak_2 \in M$-rad$(N)$. Since $ak_2 \in M$-rad(N), we have $ak_1 \in M$-rad(N), a contradiction. Suppose that $b(k_1 + k_2) = bk_1 + bk_2 \in M$-rad$(N)$. Since $bk_1 \in M$-rad(N), we have $bk_2 \in M$-rad(N), a contradiction again. Thus $aK \subseteq M$-rad(N) or $bK \subseteq M$-rad(N). □

The following theorem offers a characterization of 2-absorbing primary submodules.

Theorem 2.17. *Let M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is a 2-absorbing primary submodule of M;*
2. *If $I_1I_2K \subseteq N$ for some ideals $I_1,\ I_2$ of R and some submodule K of M, then either $I_1I_2 \subseteq (N :_R M)$ or $I_1K \subseteq M$-rad(N) or $I_2K \subseteq M$-rad(N);*
3. *If $N_1N_2N_3 \subseteq N$ for some submodules $N_1,\ N_2$ and N_3 of M, then either $N_1N_2 \subseteq N$ or $N_1N_3 \subseteq M$-rad(N) or $N_2N_3 \subseteq M$-rad(N).*

Proof. (1)⇒(2) Suppose that N is a 2-absorbing primary submodule of M and $I_1I_2K \subseteq N$ for some ideals I_1, I_2 of R and some submodule K of M and $I_1I_2 \nsubseteq (N :_R M)$. We show that $I_1K \subseteq M$-rad(N) or $I_2K \subseteq M$-rad(N). Suppose that $I_1K \nsubseteq M$-rad(N) and $I_2K \nsubseteq M$-rad(N). Then there are $a_1 \in I_1$ and $a_2 \in I_2$ such that $a_1K \nsubseteq M$-rad(N) and $a_2K \nsubseteq M$-rad(N). Since $a_1a_2K \subseteq N$ and neither $a_1K \subseteq M$-rad(N) nor $a_2K \subseteq M$-rad(N), we have $a_1a_2 \in (N :_R M)$ by Lemma 2.16.
Since $I_1I_2 \nsubseteq (N :_R M)$, we have $b_1b_2 \notin (N :_R M)$ for some $b_1 \in I_1$ and $b_2 \in I_2$. Since $b_1b_2K \subseteq N$ and $b_1b_2 \notin (N :_R M)$, we have $b_1K \subseteq M$-rad(N) or $b_2K \subseteq M$-rad(N) by Lemma 2.16. We consider three cases.

Case 1. Suppose that $b_1K \subseteq M$-rad(N) but $b_2K \nsubseteq M$-rad(N). Since $a_1b_2K \subseteq N$ and neither $b_2K \subseteq M$-rad(N) nor $a_1K \subseteq M$-rad(N), we conclude that $a_1b_2 \in (N :_R M)$ by Lemma 2.16. Since $b_1K \subseteq M$-rad(N) but $a_1K \nsubseteq M$-rad(N), we conclude that $(a_1 + b_1)K \nsubseteq M$-rad$(N)$. Since $(a_1 + b_1)b_2K \subseteq N$ and neither $b_2K \subseteq M$-rad(N) nor $(a_1 + b_1)K \subseteq M$-rad(N), we conclude that $(a_1 + b_1)b_2 \in (N :_R M)$ by Lemma 2.16. Since $(a_1 + b_1)b_2 = a_1b_2 + b_1b_2 \in (N :_R M)$ and $a_1b_2 \in (N :_R M)$, we conclude that $b_1b_2 \in (N :_R M)$, a contradiction.

Case 2. Suppose that $b_2K \subseteq M$-rad(N) but $b_1K \nsubseteq M$-rad(N). Similar to the previous case we reach to a contradiction.

Case 3. Suppose that $b_1K \subseteq M$-rad(N) and $b_2K \subseteq M$-rad(N). Since $b_2K \subseteq M$-rad(N) and $a_2K \nsubseteq M$-rad(N), we conclude that $(a_2 + b_2)K \nsubseteq M$-rad$(N)$. Since $a_1(a_2 + b_2)K \subseteq N$ and neither $a_1K \subseteq M$-rad(N) nor $(a_2 + b_2)K \subseteq M$-rad(N), we conclude that $a_1(a_2 + b_2) = a_1a_2 + a_1b_2 \in (N :_R M)$ by Lemma 2.16. Since $a_1a_2 \in (N :_R M)$ and $a_1a_2 + a_1b_2 \in (N :_R M)$, we conclude that $a_1b_2 \in (N :_R M)$. Since $b_1K \subseteq M$-rad(N) and $a_1K \nsubseteq M$-rad(N), we conclude that $(a_1 + b_1)K \nsubseteq M$-rad$(N)$. Since $(a_1 + b_1)a_2K \subseteq N$ and neither $a_2K \subseteq M$-rad(N) nor $(a_1 + b_1)K \subseteq M$-rad(N), we conclude that $(a_1 + b_1)a_2 = a_1a_2 + b_1a_2 \in (N :_R M)$ by Lemma 2.16. Since $a_1a_2 \in (N :_R M)$ and $a_1a_2 + b_1a_2 \in (N :_R M)$, we conclude that $b_1a_2 \in (N :_R M)$. Now, since $(a_1 + b_1)(a_2 + b_2)K \subseteq N$ and neither $(a_1 + b_1)K \subseteq M$-rad(N) nor $(a_2 + b_2)K \subseteq M$-rad(N), we conclude that

$(a_1 + b_1)(a_2 + b_2) = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 \in (N :_R M)$ by Lemma 2.16. Since $a_1a_2, a_1b_2, b_1a_2 \in (N :_R M)$, we have $b_1b_2 \in (N :_R M)$, a contradiction. Consequently $I_1K \subseteq M\text{-rad}(N)$ or $I_2K \subseteq M\text{-rad}(N)$.
(2)⇒(1) is trivial.
(2)⇒(3) Let $N_1N_2N_3 \subseteq N$ for some submodules $N_1,\ N_2$ and N_3 of M such that $N_1N_2 \nsubseteq N$. Since M is multiplication, there are ideals $I_1,\ I_2$ of R such that $N_1 = I_1M,\ N_2 = I_2M$. Clearly $I_1I_2N_3 \subseteq N$ and $I_1I_2 \nsubseteq (N :_R M)$. Therefore $I_1N_3 \subseteq M\text{-rad}(N)$ or $I_2N_3 \subseteq M\text{-rad}(N)$, which implies that $N_1N_3 \subseteq M\text{-rad}(N)$ or $N_2N_3 \subseteq M\text{-rad}(N)$.
(3)⇒(2) Suppose that $I_1I_2K \subseteq N$ for some ideals $I_1,\ I_2$ of R and some submodule K of M. It is sufficient to set $N_1 := I_1M$, $N_2 := I_2M$ and $N_3 = K$ in part (3). □

Theorem 2.18. *Let M be a multiplication R-module and N a submodule of M. If $(N :_R M)$ is a 2-absorbing primary ideal of R, then N is a 2-absorbing primary submodule of M.*

Proof. Let $I_1I_2K \subseteq N$ for some ideals $I_1,\ I_2$ of R and some submodule K of M. Since M is multiplication, then there is an ideal I_3 of R such that $K = I_3M$. Hence $I_1I_2I_3 \subseteq (N :_R M)$ which implies that either $I_1I_2 \subseteq (N :_R M)$ or $I_1I_3 \subseteq \sqrt{(N :_R M)}$ or $I_2I_3 \subseteq \sqrt{(N :_R M)}$, by [11, Theorem 2.19]. If $I_1I_2 \subseteq (N :_R M)$, then we are done. So, suppose that $I_1I_3 \subseteq \sqrt{(N :_R M)}$. Thus $I_1I_3M = I_1K \subseteq \sqrt{(N :_R M)}M = M\text{-rad}(N)$. Similary if $I_2I_3 \subseteq \sqrt{(N :_R M)}$, then we have $I_2K \subseteq M\text{-rad}(N)$. It completes the proof, by Theorem 2.17. □

The following example shows that Theorem 2.18 is not satisfied in general.

Example 2.19. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ and $N = 6\mathbb{Z} \times 0$ a submodule of M. Observe that $\mathbb{Z}\times 0$, $2\mathbb{Z}\times\mathbb{Z}$ and $3\mathbb{Z}\times\mathbb{Z}$ are some of the prime submodules of M containing N. Also $(N :_{\mathbb{Z}} M) = 0$ is a 2-absorbing primary ideal of $\mathbb{Z}$. On the other hand, since $2.3.(1, 0) = (6, 0) \in N$, $2.3 \notin (N :_{\mathbb{Z}} M)$, $2.(1, 0) = (2, 0) \notin M\text{-rad}(N) \subseteq (\mathbb{Z} \times 0) \cap (2\mathbb{Z} \times \mathbb{Z}) \cap (3\mathbb{Z} \times \mathbb{Z}) = 6\mathbb{Z} \times 0 = N$ and $3.(1, 0) = (3, 0) \notin M\text{-rad}(N) = N$, so N is not a 2-absorbing primary submodule of M.

Theorem 2.20. *Let M be a multiplication R-module and N_1 and N_2 be primary submodules of M. Then $N_1 \cap N_2$ is a 2-absorbing primary submodule of M. If in addition M is finitely generated faithful, then N_1N_2 is a 2-absorbing primary submodule of M.*

Proof. Since N_1 and N_2 are primary submodules of M, then $(N_1 :_R M)$ and $(N_2 :_R M)$ are primary ideals of R. Hence $(N_1 :_R M)(N_2 :_R M)$ and $(N_1 \cap N_2 :_R M) = (N_1 :_R M) \cap (N_2 :_R M)$ are 2-absorbing primary ideals of R, by [11, Theorem 2.4]. Therefore, Theorem 2.18 implies that $N_1 \cap N_2$ is a 2-absorbing primary submodule of M. If M is a finitely generated faithful multiplication R-module, then $(N_1N_2 :_R M) = (N_1 :_R M)(N_2 :_R M)$. So, again by Theorem 2.18 we deduce that N_1N_2 is a 2-absorbing primary submodule of M. □

Let M be a multiplication R-module and N a primary submodule of M. We know that $\sqrt{(N :_R M)}$ is a prime ideal of R and so $P = M\text{-rad}(N) = \sqrt{(N :_R M)}M$ is a prime submodule of M. In this case we say that N is a *P-primary submodule of M*.

Corollary 2.21. *Let M be a multiplication R-module and P_1 and P_2 be prime submodules of M. Suppose that P_1^n is a P_1-primary submodule of M for some positive integer $n \geq 1$ and P_2^m is a P_2-primary submodule of M for some positive integer $m \geq 1$.*

1. *$P_1^n \cap P_2^m$ is a 2-absorbing primary submodule of M.*
2. *If in addition M is finitely generated faithful, then $P_1^n P_2^m$ is a 2-absorbing primary submodule of M.*

Theorem 2.22. *Let M be a multiplication R-module and N be a submodule of M that has a primary decomposition. If M-rad$(N) = \mathfrak{M}_1 \cap \mathfrak{M}_2$ where $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two maximal submodules of M, then N is a 2-absorbing primary submodule of M.*

Proof. Assume that $N = N_1 \cap \cdots \cap N_n$ is a primary decomposition. By Proposition 2.14(3), $M\text{-rad}(N) = M\text{-rad}(N_1) \cap \cdots \cap M\text{-rad}(N_n) = \mathfrak{M}_1 \cap \mathfrak{M}_2$. Since M-rad(N_i)'s are prime submodules of M, then $\{M\text{-rad}(N_1), \ldots, M\text{-rad}(N_n)\} = \{\mathfrak{M}_1, \mathfrak{M}_2\}$, by [3, Theorem 3.16]. Without loss of generality we may assume that for some $1 \leq t < n$, $\{M\text{-rad}(N_1), \ldots, M\text{-rad}(N_t)\} = \{\mathfrak{M}_1\}$ and $\{M\text{-rad}(N_{t+1}), \ldots, M\text{-rad}(N_n)\} = \{\mathfrak{M}_2\}$. Set $K_1 := N_1 \cap \cdots \cap N_t$ and $K_2 := N_{t+1} \cap \cdots \cap N_n$. By [8, Lemma 1.2.2], K_1 is an $\mathfrak{M}_1$-primary submodule and K_2 is an $\mathfrak{M}_2$-primary submodule of M. Therefore, by Theorem 2.20, $N = K_1 \cap K_2$ is 2-absorbing primary. □

Lemma 2.23. *([22, Corollary 1.3]) Let M and M' be R-modules with $f : M \to M'$ an R-module epimorphism. If N is a submodule of M containing $Ker(f)$, then $f(M\text{-}rad(N)) = M'\text{-}rad(f(N))$.*

Theorem 2.24. *Let $f : M \to M'$ be a homomorphism of R-modules.*

1. *If N' is a 2-absorbing primary submodule of M', then $f^{-1}(N')$ is a 2-absorbing primary submodule of M.*

2. *If f is epimorphism and N is a 2-absorbing primary submodule of M containing $Ker(f)$, then $f(N)$ is a 2-absorbing primary submodule of M'.*

Proof. (1) Let $a, b \in R$ and $m \in M$ such that $abm \in f^{-1}(N')$. Then $abf(m) \in N'$. Hence $ab \in (N' :_R M')$ or $af(m) \in M'$-rad(N') or $bf(m) \in M'$-rad(N'), and thus $ab \in (f^{-1}(N') :_R M)$ or $am \in f^{-1}(M'$-rad$(N'))$ or $bm \in f^{-1}(M'$-rad$(N'))$. By using the inclusion $f^{-1}(M'$-rad$(N')) \subseteq M$-rad$(f^{-1}(N'))$, we conclude that $f^{-1}(N')$ is a 2-absorbing primary submodule of M.
(2) Let $a, b \in R$, $m' \in M'$ and $abm' \in f(N)$. By assumption there exists $m \in M$ such that $m' = f(m)$ and so $f(abm) \in f(N)$. Since $Ker(f) \subseteq N$, we have $abm \in N$. It implies that $ab \in (N :_R M)$ or $am \in M$-rad(N) or $bm \in M$-rad(N). Hence $ab \in (f(N) :_R M')$ or $am' \in f(M$-rad$(N)) = M'$-rad$(f(N))$ or $bm' \in f(M$-rad$(N)) = M'$-rad$(f(N))$. Consequently $f(N)$ is a 2-absorbing primary submodule of M'. □

As an immediate consequence of Theorem 2.24(2) we have the following Corollary.

Corollary 2.25. *Let M be an R-module and $L \subseteq N$ be submodules of M. If N is a 2-absorbing primary submodule of M, then N/L is a 2-absorbing primary submodule of M/L.*

Theorem 2.26. *Let K and N be submodules of M with $K \subset N \subset M$. If K is a 2-absorbing primary submodule of M and N/K is a weakly 2-absorbing primary submodule of M/K, then N is a 2-absorbing primary submodule of M.*

Proof. Let $a, b \in R$, $m \in M$ and $abm \in N$. If $abm \in K$, then $am \in M$-rad$(K) \subseteq M$-rad(N) or $bm \in M$-rad$(K) \subseteq M$-rad(N) or $ab \in (K :_R M) \subseteq (N :_R M)$ as it is needed.
So suppose that $abm \notin K$. Then $0 \neq ab(m + K) \in N/K$ that implies, $a(m + K) \in M/K$-rad$(N/K) = \frac{M-\mathrm{rad}(N)}{K}$ or $b(m + K) \in M/K$-rad(N/K) or $ab \in (N/K :_R M/K)$. It means that $am \in M$-rad(N) or $bm \in M$-rad(N) or $ab \in (N :_R M)$, which completes the proof. □

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is of the form $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2. In addition, if M_i is a multiplication R_i-module, for $i = 1, 2$, then M is a multiplication R-module. In this case, for each submodule $N = N_1 \times N_2$ of M we have M-rad$(N) = M_1$-rad$(N_1) \times M_2$-rad(N_2).

Theorem 2.27. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ where M_1 is a multiplication R_1-module and M_2 is a multiplication R_2-module.*

1. *A proper submodule K_1 of M_1 is a 2-absorbing primary submodule if and only if $N = K_1 \times M_2$ is a 2-absorbing primary submodule of M.*
2. *A proper submodule K_2 of M_2 is a 2-absorbing primary submodule if and only if $N = M_1 \times K_2$ is a 2-absorbing primary submodule of M.*
3. *If K_1 is a primary submodule of M_1 and K_2 is a primary submodule of M_2, then $N = K_1 \times K_2$ is a 2-absorbing primary submodule of M.*

Proof. (1) Suppose that $N = K_1 \times M_2$ is a 2-absorbing primary submodule of M. From our hypothesis, N is proper, so $K_1 \neq M_1$. Set $M' = \frac{M}{\{0\}\times M_2}$. Hence $N' = \frac{N}{\{0\}\times M_2}$ is a 2-absorbing primary submodule of M' by Corollary 2.25. Also observe that $M' \cong M_1$ and $N' \cong K_1$. Thus K_1 is a 2-absorbing primary submodule of M_1. Conversely, if K_1 is a 2-absorbing primary submodule of M_1, then it is clear that $N = K_1 \times M_2$ is a 2-absorbing primary submodule of M.
(2) It can be easily verified similar to (1).
(3) Assume that $N = K_1 \times K_2$ where K_1 and K_2 are primary submodules of M_1 and M_2, respectively. Hence $(K_1 \times M_2) \cap (M_1 \times K_2) = K_1 \times K_2 = N$ is a 2-absorbing primary submodule of M, by parts (1) and (2) and Theorem 2.20. □

Theorem 2.28. *Let $R = R_1 \times R_2$ and $M = M_1 \times M_2$ be a finitely generated multiplication R-module where M_1 is a multiplication R_1-module and M_2 is a multiplication R_2-module. If $N = N_1 \times N_2$ is a proper submodule of M, then the followings are equivalent.*

1. *N is a 2-absorbing primary submodule of M.*
2. *$N_1 = M_1$ and N_2 is a 2-absorbing primary submodule of M_2 or $N_2 = M_2$ and N_1 is a 2-absorbing primary submodule of M_1 or N_1, N_2 are primary submodules of M_1, M_2, respectively.*

Proof. (1)⇒(2) Suppose that $N = N_1 \times N_2$ is a 2-absorbing primary submodule of M. Then $(N : M) = (N_1 : M_1) \times (N_2 : M_2)$ is a 2-absorbing primary ideal of $R = R_1 \times R_2$ by Theorem 2.6. From Theorem 2.3 in [11], we have $(N_1 : M_1) = R_1$ and $(N_2 : M_2)$ is a 2-absorbing primary ideal of R_2 or $(N_2 : M_2) = R_2$ and $(N_1 : M_1)$ is a 2-absorbing primary ideal of R_1 or $(N_1 : M_1)$ and $(N_2 : M_2)$ are primary ideals of R_1, R_2, respectively. Assume that $(N_1 : M_1) = R_1$ and $(N_2 : M_2)$ is a 2-absorbing primary ideal of R_2.

Thus $N_1 = M_1$ and N_2 is a 2-absorbing primary submodule of M_2 by Theorem 2.18. Similarly if $(N_2 : M_2) = R_2$ and $(N_1 : M_1)$ is a 2-absorbing primary ideal of R_1, then $N_2 = M_2$ and N_1 is a 2-absorbing primary submodule of M. And if the last case hold, then clearly we conclude that N_1, N_2 are primary submodules of M_1, M_2, respectively.
(2)⇒(1) It is clear from Theorem 2.27. □

References

[1] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra, **36** (2008), 4620–4642.

[2] M. M. Ali, *Invertiblity of Multiplication Modules III*, New Zeland J. Math., **39** (2009), 193-213.

[3] R. Ameri, *On the prime submodules of multiplication modules*, Inter. J. Math. Math. Sci., **27** (2003), 1715–1724.

[4] D. D. Anderson and M. Batanieh, *Generalizations of prime ideals*, Comm. Algebra, **36** (2008), 686–696.

[5] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math., **29** (2003), 831–840.

[6] D. F. Anderson and A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra, **39** (2011), 1646–1672.

[7] F. Anderson and K. Fuller, *Rings and categories of modules*. New-York: Springer-Verlag, 1992.

[8] R. B. Ash, *A course in commutative algebra*. University of Illinois, 2006.

[9] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., **75** (2007), 417–429.

[10] A. Badawi and A. Yousefian Darani, *On weakly 2-absorbing ideals of commutative rings*, Houston J. Math., **39** (2013), 441–452.

[11] A. Badawi, Ü. Tekir and E. Yetkin, *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc., **51** (4) (2014), 1163–1173.

[12] A. Barnard, *Multiplication modules*, J. Algebra, **71** (1981), 174–178.

[13] M. Behboodi and H. Koohi, *Weakly prime modules*, Vietnam J. Math., **32** (2) (2004), 185–195.

[14] S. M. Bhatwadekar and P. K. Sharma, *Unique factorization and birth of almost primes*, Comm. Algebra, **33** (2005), 43–49.

[15] S. Ebrahimi Atani, F. Çallialp and Ü. Tekir, *A Short note on the primary submodules of multiplication modules*, Inter. J. Algebra, **8** (1) (2007), 381–384.

[16] S. Ebrahimi Atani and F. Farzalipour, *On weakly prime submodules*, Tamk. J. Math., **38** (3) (2007), 247–252.

[17] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, **16** (1988), 755–779.

[18] C. Faith, *Algebra: Rings, modules and categories I*, Springer-Verlag Berlin Heidelberg New York 1973.

[19] R. Gilmer, *Multiplicative ideal theory*. Queens Papers Pure Appl. Math. 90, Queens University, Kingston (1992).

[20] M. Khoramdel and S. Dolati Pish Hesari, *Some notes on Dedekind modules*, Hacettepe J. Math. Stat., **40** (2011), 627–634.

[21] R. L. McCasland and M. E. Moore, *On radicals of submodules of finitely generated modules*, Canad. Math. Bull., **29** (1986), 37–39.

[22] R. L. McCasland and M. E. Moore, *Radicals of submodules*, Comm. Algebra, **19** (1991), 1327-1341.

[23] A. G. Naoum and F. H. Al-Alwan, *Dedekind modules*, Comm. Algebra, **24** (2)(1996), 397-412.

[24] Sh. Payrovi and S. Babaei, *On* 2-*absorbing submodules*, Algebra Colloq., **19** (2012), 913-920.

[25] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math., **50** (1988), 223–235.

[26] A. Yousefian Darani and F. Soheilnia, *On* 2-*absorbing and weakly* 2-*absorbing submodules*, Thai J. Math., **9** (2011), 577-584

[27] A. Yousefian Darani and F. Soheilnia, *On n-absorbing submodules*, Math. Commun., **17** (2012), 547-557.

Hojjat Mostafanasab,
Department of Mathematics and Applications,
University of Mohaghegh Ardabili,
P. O. Box 179, Ardabil, Iran.
Email: h.mostafanasab@uma.ac.ir

Ece Yetkin,
Department of Mathematics,
Marmara University,
Ziverbey, Goztepe, Istanbul 34722, Turkey.
Email: ece.yetkin@marmara.edu.tr

Ünsal Tekir,
Department of Mathematics,
Marmara University,
Ziverbey, Goztepe, Istanbul 34722, Turkey.
Email: utekir@marmara.edu.tr

Ahmad Yousefian Darani,
Department of Mathematics and Applications,
University of Mohaghegh Ardabili,
P. O. Box 179, Ardabil, Iran.
Email: yousefian@uma.ac.ir