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Inversion of Weinstein intertwining operator and its dual using Weinstein wavelets

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Abstract

In this paper, we consider the Weinstein intertwining operator $\mathcal{R}_W^{\alpha,d}$ and its dual ${}^t\mathcal{R}_W^{\alpha,d}$. Using these operators, we give relations between the Weinstein and the classical continuous wavelet transforms. Finally, using the Weinstein continuous wavelet transform, we deduce the formulas which give the inverse operators of $\mathcal{R}_W^{\alpha,d}$ and ${}^t\mathcal{R}_W^{\alpha,d}$.

1 INTRODUCTION

In this paper, we consider the Weinstein operator $\Delta_W^{\alpha,d}$ defined on $\mathbb{R}_+^{d+1} = \mathbb{R}^d \times]0, +\infty[$, by:

$$\Delta_W^{\alpha,d} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_\alpha, \ \alpha > -\frac{1}{2}, \tag{1}$$

where Δ_d is the Laplacian for the d first variables and L_α is the Bessel operator for the last variable defined on $]0, +\infty[$ by :

$$L_\alpha u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha+1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}} = \frac{1}{x_{d+1}^{2\alpha+1}} \frac{\partial}{\partial x_{d+1}} \left[x_{d+1}^{2\alpha+1} \frac{\partial u}{\partial x_{d+1}} \right].$$

The Weinstein operator $\Delta_W^{\alpha,d}$, mostly referred to as the Laplace-Bessel differential operator is now known as an important operator in analysis, due its

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applications in pure and applied Mathematics, especially in Fluid Mechanics [7].
The relevant harmonic analysis associated with the Bessel differential operator L_α goes back to S. Bochner, J. Delsarte, B. M. Levitan and has been studied by many other authors such as J. Löfström and J. peetre [11], I. Kipriyanov [9], K. Stempak [14], K. Trimèche [15], I. A. Aliev and B. Rubin [1].

The Weinstein intertwining operator is the operator $\mathcal{R}_W^{\alpha,d}$ defined on $\mathcal{E}_*(\mathbb{R}^{d+1})$ (the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable) by

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathcal{R}_W^{\alpha,d}(f)(x) = a_\alpha \int_0^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} f(x', tx_{d+1})dt,$$

where a_α is the constant given by :

$$a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}. \tag{2}$$

The dual of the operator $\mathcal{R}_W^{\alpha,d}$ is defined on $\mathcal{D}_*(\mathbb{R}^{d+1})$ (the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable) is given for all $y = (y', y_{d+1}) \in \mathbb{R}_+^{d+1}$ by :

$${}^t\mathcal{R}_W^{\alpha,d}(f)(y) = a_\alpha \int_{y_{d+1}}^{+\infty} \left(s^2 - y_{d+1}^2\right)^{\alpha-\frac{1}{2}} f(y', s)sds.$$

$\mathcal{R}_W^{\alpha,d}$ and ${}^t\mathcal{R}_W^{\alpha,d}$ are topological ismorphism on $\mathcal{E}_*(\mathbb{R}^{d+1})$ and $\mathcal{D}_*(\mathbb{R}^{d+1})$, respectively, and they serve as transmutation operators between the Weinstein operator and the Laplacian operator on $\mathbb{R}^{d+1}$.

The main objective of this paper is to establish the inverse operators of the operators $\mathcal{R}_W^{\alpha,d}$ and ${}^t\mathcal{R}_W^{\alpha,d}$ in new functional spaces, using some pseudo-differential operators and using Weinstein wavelets.
For this purpose, we define and study the Weinstein continuous wavelet transform S_g^W and we establish for this transform a Plancherel and an inversion formulas. Using the operator $\mathcal{R}_W^{\alpha,d}$ and its dual ${}^t\mathcal{R}_W^{\alpha,d}$, we give relations between S_g^W and the classical continuous wavelet transform S_g. Using the inversion formulas of the transforms S_g^W and S_g, we deduce the formulas which give the inverse operators of $\mathcal{R}_W^{\alpha,d}$ and ${}^t\mathcal{R}_W^{\alpha,d}$, using wavelets.
The contents of this paper is as follows :
In the second Section, we recall some basic harmonic analysis results related with the Weinstein operator developed in [2], [3] and [4].
In the third Section, we list some basic properties of the Weinstein intertwining operator $\mathcal{R}_W^{\alpha,d}$ and its dual ${}^t\mathcal{R}_W^{\alpha,d}$.

In the fourth Section, we define and characterize new spaces of functions $\mathcal{S}_{*,0}(\mathbb{R}^{d+1})$ and $\mathcal{S}^0_{*,0}(\mathbb{R}^{d+1})$ on which the operator $\mathcal{R}^{\alpha,d}_W$ and its dual ${}^t\mathcal{R}^{\alpha,d}_W$ are bijective.
In the fifth Section, we give the definition of the Weinstein continuous wavelet transform S^W_g and we establish a Plancherel and an inversion formulas. Using the operator $\mathcal{R}^{\alpha,d}_W$ and its dual ${}^t\mathcal{R}^{\alpha,d}_W$, we give relations between S^W_g and the classical continuous wavelet transform S_g. Finally, in the last Section, using the inversion formulas of the transforms S^W_g and S_g, we deduce the formulas which give the inverse operators of $\mathcal{R}^{\alpha,d}_W$ and ${}^t\mathcal{R}^{\alpha,d}_W$.

2 Harmonic analysis associated with the Weinstein-Laplace operator

In this section, we shall collect some results and definitions from the theory of the harmonic analysis associated with the Weinstein operator $\Delta^{\alpha,d}_W$ defined on $\mathbb{R}^{d+1}_+$ by the relation (1). Main references are ([2, 3, 4, 5, 8, 12, 13]).

Let us begin by the following result, which gives the eigenfunction $\Psi^{\alpha,d}_\lambda$ of the Weinstein operator $\Delta^{\alpha,d}_W$.

Proposition 1.
For all $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{d+1}) \in \mathbb{R}^{d+1}_+$, the system

$$\left\{ \begin{array}{c} \frac{\partial^2 u}{\partial x_j^2}(x) = -\lambda_j^2 u(x), \ \ if \quad 1 \le j \le d \\ L_\alpha u(x) = -\lambda^2_{d+1} u(x), \\ u(0) = 1, \ \frac{\partial u}{\partial x_{d+1}}(0) = 0 \ and \ \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \ \ if \quad 1 \le j \le d \end{array} \right. \tag{3}$$

has a unique solution $\Psi^{\alpha,d}_\lambda$ given by

$$\forall z \in \mathbb{C}^{d+1}, \ \Psi^{\alpha,d}_\lambda(z) = e^{-i\langle z', \ \lambda'\rangle} j_\alpha(\lambda_{d+1} z_{d+1}), \tag{4}$$

where $z = (z', x_{d+1})$, $z' = (z_1, z_2, ..., z_d)$ and j_α is the normalized Bessel function of index α, defined by :

$$\forall \xi \in \mathbb{C}, \ j_\alpha(\xi) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} (\frac{\xi}{2})^{2n}.$$

Remark 1. *The Weinstein kernel $\Lambda_{\alpha,d} : (\lambda, z) \mapsto \Psi^{\alpha,d}_\lambda(z)$ has a unique extention to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ and can be written in the form :*

$$\forall x, y \in \mathbb{C}^{d+1}, \Lambda_{\alpha,d}(x, y) = a_\alpha e^{-i\langle x', \ y'\rangle} \int_0^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} cos(t x_{d+1} y_{d+1}) dt, \tag{5}$$

where $x=(x',x_{d+1})$, $x'=(x_1,x_2,...,x_d)$ *and* a_α *is the constant given by the relation (2).*

The following result summarizes some of the Weinstein kernel's properties.

Proposition 2.
i) For all $\lambda,\ z\in\mathbb{C}^{d+1}$ *and* $t\in\mathbb{R}$*, we have*

$$\Lambda_{\alpha,d}(\lambda,0)=1,\ \Lambda_{\alpha,d}(\lambda,z)=\Lambda_{\alpha,d}(z,\lambda)\ \text{and}\ \Lambda_{\alpha,d}(\lambda,tz)=\Lambda_{\alpha,d}(t\lambda,z).$$

ii) For all $\nu\in\mathbb{N}^{d+1}$, $x\in\mathbb{R}_+^{d+1}$ *and* $z\in\mathbb{C}^{d+1}$*, we have*

$$|D_z^\nu\Lambda_{\alpha,d}(x,z)|\le\|x\|^{|\nu|}\exp(\|x\|\,\|\operatorname{Im}z\|), \tag{6}$$

where $D_z^\nu=\frac{\partial^\nu}{\partial z_1^{\nu_1}...\partial z_{d+1}^{\nu_{d+1}}}$ *and* $|\nu|=\nu_1+...+\nu_{d+1}$*. In particular*

$$\forall x,y\in\mathbb{R}_+^{d+1},\ |\Lambda_{\alpha,d}(x,y)|\le 1. \tag{7}$$

Notations. In what follows, we need the following notations:
- $C_*(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $C_{*,c}(\mathbb{R}^{d+1})$, the space of continuous functions on $\mathbb{R}^{d+1}$ with compact support, even with respect to the last variable.
- $C_*^p(\mathbb{R}^{d+1})$, the space of functions of class C^p on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{E}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{S}_*(\mathbb{R}^{d+1})$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^{d+1}$, even with respect to the last variable.
- $\mathcal{D}_*(\mathbb{R}^{d+1})$, the space of C^∞-functions on $\mathbb{R}^{d+1}$ which are of compact support, even with respect to the last variable.
- $L_\alpha^p(\mathbb{R}_+^{d+1})$, $1\le p\le+\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$ such that

$$\|f\|_{\alpha,\infty}=\operatorname*{ess\,sup}_{x\in\mathbb{R}_+^{d+1}}|f(x)|<+\infty,$$

$$\|f\|_{\alpha,p}=\left[\int_{\mathbb{R}_+^{d+1}}|f(x)|^p d\mu_{\alpha,d}(x)\right]^{\frac{1}{p}}<+\infty,\ \text{if } 1\le p<+\infty,$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}_+^{d+1}$ given by :

$$d\mu_{\alpha,d}(x)=C_{\alpha,d}x_{d+1}^{2\alpha+1}dx, \tag{8}$$

dx is the Lebesgue measure on $\mathbb{R}^{d+1}$ and $C_{\alpha,d}$ is the constant given by

$$C_{\alpha,d} = \frac{1}{(2\pi)^{\frac{d}{2}} 2^{\alpha}\Gamma(\alpha+1)}. \tag{9}$$

• $\mathcal{H}_*(\mathbb{C}^{d+1})$, the space of entire functions on $\mathbb{C}^{d+1}$, even with respect to the last variable, rapidly decreasing and of exponential type.

Definition 1. *The Weinstein transform is given for $f \in L^1_\alpha(\mathbb{R}^{d+1}_+)$ by*

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \mathcal{F}^{\alpha,d}_W(f)(\lambda) = \int_{\mathbb{R}^{d+1}_+} f(x)\Lambda_{\alpha,d}(x,\lambda)d\mu_{\alpha,d}(x), \tag{10}$$

where $\mu_{\alpha,d}$ is the measure on $\mathbb{R}^{d+1}_+$ given by the relation (8).

Using the properties of the classical Fourier transform on $\mathbb{R}^d$ and of the Bessel transform, one can easily see the following relation, which will play an important role in the sequel.

Example 1. *Let ϕ_t, $t > 0$, be the function defined by :*

$$\forall x \in \mathbb{R}^{d+1},\ \phi_t(x) = \frac{1}{(2t)^{\alpha+\frac{d}{2}+1}} e^{-\frac{\|x\|^2}{4t}}. \tag{11}$$

Then the Weinstein transform $\mathcal{F}^{\alpha,d}_W$ of ϕ_t is given by :

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \mathcal{F}^{\alpha,d}_W(\phi_t)(\lambda) = e^{-t\|\lambda\|^2}.$$

Some basic properties of the transform $\mathcal{F}^{\alpha,d}_W$ are summarized in the following results. For the proofs, we refer to [4, 5, 6].

Proposition 3.
i) For all $f \in L^1_\alpha(\mathbb{R}^{d+1}_+)$, we have

$$\|\mathcal{F}^{\alpha,d}_W(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,1}. \tag{12}$$

ii) For $m \in \mathbb{N}$ and $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have

$$\forall y \in \mathbb{R}^{d+1}_+,\ \mathcal{F}^{\alpha,d}_W\left[\left(\triangle^{\alpha,d}_W\right)^m f\right](y) = (-1)^m \|y\|^{2m}\mathcal{F}^{\alpha,d}_W(f)(y). \tag{13}$$

iii) For all f in $\mathcal{S}_*(\mathbb{R}^{d+1})$ and $m \in \mathbb{N}$, we have

$$\forall \lambda \in \mathbb{R}^{d+1}_+,\ \left(\triangle^{\alpha,d}_W\right)^m \left[\mathcal{F}^{\alpha,d}_W(f)\right](\lambda) = \mathcal{F}^{\alpha,d}_W(P_m f)(\lambda), \tag{14}$$

where $P_m(\lambda) = (-1)^m \|\lambda\|^{2m}$.

Theorem 1.
i) The Weinstein transform $\mathcal{F}_W^{\alpha,d}$ is a topological isomorphism from $\mathcal{S}_*(\mathbb{R}^{d+1})$ onto itself and from $\mathcal{D}_*(\mathbb{R}^{d+1})$ onto $\mathcal{H}_*(\mathbb{C}^{d+1})$.
ii) Let $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$. The inverse transform $\left(\mathcal{F}_W^{\alpha,d}\right)^{-1}$ is given by

$$\forall x \in \mathbb{R}_+^{d+1},\ \left(\mathcal{F}_W^{\alpha,d}\right)^{-1}(f)(x) = \mathcal{F}_W^{\alpha,d}(f)\,(-x)\,. \tag{15}$$

iii) Let $f \in L_\alpha^1(\mathbb{R}_+^{d+1})$. If $\mathcal{F}_W^{\alpha,d}(f) \in L_\alpha^1(\mathbb{R}_+^{d+1})$, then we have

$$f(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)\,(y)\,\Lambda_{\alpha,d}(-x,y)d\mu_{\alpha,d}(y),\ a.e\ x \in \mathbb{R}_+^{d+1}. \tag{16}$$

Theorem 2.
i) For all $f, g \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have the following Parseval formula

$$\int_{\mathbb{R}_+^{d+1}} f(x)\overline{g(x)}d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} \mathcal{F}_W^{\alpha,d}(f)(\lambda)\overline{\mathcal{F}_W^{\alpha,d}(g)(\lambda)}d\mu_{\alpha,d}(\lambda). \tag{17}$$

ii) (Plancherel formula).
For all $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$, we have :

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2\, d\mu_{\alpha,d}(x) = \int_{\mathbb{R}_+^{d+1}} \left|\mathcal{F}_W^{\alpha,d}(f)(\lambda)\right|^2 d\mu_{\alpha,d}(\lambda). \tag{18}$$

iii) (Plancherel Theorem) :
The transform $\mathcal{F}_W^{\alpha,d}$ extends uniquely to an isometric isomorphism on $L_\alpha^2(\mathbb{R}_+^{d+1})$.

Definition 2. *The translation operator T_x, $x \in \mathbb{R}_+^{d+1}$, associated with the Weinstein operator $\Delta_W^{\alpha,d}$ is defined on $C_*(\mathbb{R}^{d+1})$, for all $y \in \mathbb{R}_+^{d+1}$, by :*

$$T_x f\,(y) = \frac{a_\alpha}{2}\int_0^\pi f\left(x' + y',\ \sqrt{x_{d+1}^2 + y_{d+1}^2 + 2x_{d+1}y_{d+1}\cos\theta}\right)(\sin\theta)^{2\alpha}\, d\theta, \tag{19}$$

where $x' + y' = (x_1 + y_1, ..., x_d + y_d)$ and a_α is the constant given by (2).

Example 2. *Let ϕ_t, $t > 0$, be the function given by the relation (11). Then for all $x,\ y \in \mathbb{R}_+^{d+1}$, we have*

$$T_x\,(\phi_t)\,(y) = \frac{1}{(2t)^{\alpha+\frac{d}{2}+1}} e^{-\frac{\|x\|^2+\|y\|^2}{4t}} \Lambda_{\alpha,d}(x, -i\frac{y}{2t}). \tag{20}$$

The following proposition summarizes some properties of the Weinstein translation operator.

Proposition 4.
i) For $f \in C_*(\mathbb{R}^{d+1})$, we have

$$\forall x,\ y \in \mathbb{R}_+^{d+1},\ T_x f\,(y) = T_y f\,(x) \ \text{ and } T_0 f = f.$$

ii) For all $f \in \mathcal{E}_*(\mathbb{R}^{d+1})$ and $y \in \mathbb{R}_+^{d+1}$, the function $x \mapsto T_x f\,(y)$ belongs to $\mathcal{E}_*(\mathbb{R}^{d+1})$.
iii) We have

$$\forall x \in \mathbb{R}_+^{d+1},\ \Delta_W^{\alpha,d} \circ T_x = T_x \circ \Delta_W^{\alpha,d}.$$

iv) Let $f \in L^p_\alpha(\mathbb{R}_+^{d+1})$, $1 \le p \le +\infty$ and $x \in \mathbb{R}_+^{d+1}$. Then $T_x f$ belongs to $L^p_\alpha(\mathbb{R}_+^{d+1})$ and we have

$$\|T_x f\|_{\alpha,p} \le \|f\|_{\alpha,p}.$$

v) The function $\Lambda_{\alpha,d}\,(.,\lambda)$, $\lambda \in \mathbb{C}^{d+1}$, satisfies on $\mathbb{R}_+^{d+1}$ the following product formula:

$$\forall y \in \mathbb{R}_+^{d+1},\ \Lambda_{\alpha,d}\,(x,\lambda)\,\Lambda_{\alpha,d}\,(y,\lambda) = T_x\left[\Lambda_{\alpha,d}\,(.,\lambda)\right](y)\,. \tag{21}$$

vi) Let $f \in L^p_\alpha(\mathbb{R}_+^{d+1})$, $p = 1$ or 2 and $x \in \mathbb{R}_+^{d+1}$, we have

$$\forall y \in \mathbb{R}_+^{d+1},\ \mathcal{F}_W^{\alpha,d}\,(T_x f)\,(y) = \Lambda_{\alpha,d}\,(x,y)\,\mathcal{F}_W^{\alpha,d}\,(f)\,(y)\,. \tag{22}$$

vii) The space $\mathcal{S}_*(\mathbb{R}^{d+1})$ is invariant under the operators T_x, $x \in \mathbb{R}_+^{d+1}$.

Definition 3. *The Weinstein convolution product of $f, g \in L^1_\alpha(\mathbb{R}_+^{d+1})$ is given by:*

$$\forall x \in \mathbb{R}_+^{d+1},\ f *_W g\,(x) = \int_{\mathbb{R}_+^{d+1}} T_x f\,(-y)\,g\,(y)\,d\mu_{\alpha,d}(y). \tag{23}$$

Proposition 5.
i) Let $p, q, r \in [1,\ +\infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$.
Then for all $f \in L^p_\alpha(\mathbb{R}_+^{d+1})$ and $g \in L^q_\alpha(\mathbb{R}_+^{d+1})$, the function $f *_W g$ belongs to $L^r_\alpha(\mathbb{R}_+^{d+1})$ and we have

$$\|f *_W g\|_{\alpha,r} \le \|f\|_{\alpha,p}\|g\|_{\alpha,q}. \tag{24}$$

ii) For all $f, g \in L^1_\alpha(\mathbb{R}_+^{d+1})$, $\big(\text{resp. } \mathcal{S}_*(\mathbb{R}^{d+1})\big)$, $f *_W g \in L^1_\alpha(\mathbb{R}_+^{d+1})$ $\big(\text{resp. } \mathcal{S}_*(\mathbb{R}^{d+1})\big)$ and we have

$$\mathcal{F}_W^{\alpha,d}(f *_W g) = \mathcal{F}_W^{\alpha,d}(f)\mathcal{F}_W^{\alpha,d}(g). \tag{25}$$

3 The Weinstein intertwining operator and its dual

Definition 4.
i) The Weinstein intertwining operator is the operator $\mathcal{R}_W^{\alpha,d}$ *defined on* $C_*(\mathbb{R}^{d+1})$ *by :*

$$\forall x \in \mathbb{R}_+^{d+1},\ \mathcal{R}_W^{\alpha,d}(f)(x) = a_\alpha \int_0^1 \left(1-t^2\right)^{\alpha-\frac{1}{2}} f(x', tx_{d+1})dt, \tag{26}$$

where a_α *is the constant given by the relation (2)*
ii) The dual of the Weinstein intertwining operator is defined on $\mathcal{D}_*(\mathbb{R}^{d+1})$*, for all* $y=(y', y_{d+1}) \in \mathbb{R}_+^{d+1}$ *by :*

$$^t\mathcal{R}_W^{\alpha,d}(f)(y) = a_\alpha \int_{y_{d+1}}^{+\infty} \left(s^2 - y_{d+1}^2\right)^{\alpha-\frac{1}{2}} f(y', s)sds. \tag{27}$$

Remark 2. *For all* $x=(x', x_{d+1}), y=(y', y_{d+1}) \in \mathbb{R}_+^{d+1}$*, we have*

$$\mathcal{R}_W^{\alpha,d}(e^{-i\langle x', .\rangle} \cos(x_{d+1}.))(y) = \Lambda_{\alpha,d}(x,y). \tag{28}$$

Proposition 6.
i) $\mathcal{R}_W^{\alpha,d}$ *is a topological isomorphism from* $\mathcal{E}_*(\mathbb{R}_+^{d+1})$ *onto itself satisfying the following transmutation relation*

$$\triangle_W^{\alpha,d}(\mathcal{R}_W^{\alpha,d} f) = \mathcal{R}_W^{\alpha,d}(\triangle_{d+1} f),\ f \in \mathcal{E}_*(\mathbb{R}_+^{d+1}), \tag{29}$$

where $\triangle_{d+1} = \sum_{i=1}^{d+1} \frac{\partial^2}{\partial x_i^2}$ *is the Laplacian operator on* $\mathbb{R}^{d+1}$.
ii) $^t\mathcal{R}_W^{\alpha,d}$ *can be extended to a topological isomorphism from* $\mathcal{S}_*(\mathbb{R}^{d+1})$ *onto itself and satisfies the following transmutation relation*

$$^t\mathcal{R}_W^{\alpha,d}(\triangle_W^{\alpha,d} f) = \triangle_{d+1}(\, ^t\mathcal{R}_W^{\alpha,d} f),\ f \in \mathcal{S}_*(\mathbb{R}^{d+1}). \tag{30}$$

Example 3. *Let* $\beta \in \mathbb{R},\ \beta > 0$ *and* f_β *be the function defined on* $\mathbb{R}_+^{d+1}$ *by*

$$\forall x = (x', x_{d+1}) \in \mathbb{R}_+^{d+1},\ f_\beta(x) = \frac{\sqrt{\pi}\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma\left(\beta+\frac{1}{2}\right)} x_{d+1}^{2\beta}.$$

Then, we have

$$\mathcal{R}_W^{\alpha,d}(f_\beta) = f_\beta.$$

Proposition 7. *For all* $f \in \mathcal{S}_*\left(\mathbb{R}^{d+1}\right)$*, we have*

$$\forall y \in \mathbb{R}_+^{d+1},\ \mathcal{F}_W^{\alpha,d}(f)(y) = \mathcal{F}_o \circ\ ^t\mathcal{R}_W^{\alpha,d}(f)(y), \tag{31}$$

where $\mathcal{F}_o$ is the classical Fourier transform defined for $f \in C_{*,c}(\mathbb{R}^{d+1})$ by

$$\forall\, y \in \mathbb{R}_+^{d+1},\ \mathcal{F}_o(f)(y) = C_{\alpha,d} \int_{\mathbb{R}_+^{d+1}} f(x) e^{-i\langle y', x'\rangle} \cos(x_{d+1} y_{d+1}) dx \tag{32}$$

and $C_{\alpha,d}$ is the constant given by the relation (9).

Proposition 8. *For all $f \in \mathcal{S}_*(\mathbb{R}^{d+1})$ and $g \in \mathcal{D}_*(\mathbb{R}^{d+1})$, we have*

$$^t\,\mathcal{R}_W^{\alpha,d}(f *_W g) = {}^t\mathcal{R}_W^{\alpha,d}(f) * {}^t\mathcal{R}_W^{\alpha,d}(g), \tag{33}$$

where $*$ is the clssical convolution product given by :

$$\forall x \in \mathbb{R}^{d+1}, f * g(x) = \int_{\mathbb{R}^{d+1}} f(x-y) g(y) dy. \tag{34}$$

Proposition 9. *The operators $\mathcal{R}_W^{\alpha,d}$ and ${}^t\mathcal{R}_W^{\alpha,d}$ are linked by the following relation : for $f \in \mathcal{C}_*(\mathbb{R}^{d+1})$ and $g \in \mathcal{D}_*(\mathbb{R}^{d+1})$,*

$$\int_{\mathbb{R}_+^{d+1}} \mathcal{R}_W^{\alpha,d}(f)(y) g(x) x_{d+1}^{2\alpha+1} dx = \int_{\mathbb{R}_+^{d+1}} {}^t\mathcal{R}_W^{\alpha,d}(g)(y) f(y) dy. \tag{35}$$

4 Inversion formulas for the Weinstein intertwining operator and its dual

In this section, we show that the Weinstein intertwining operator and its dual are bijective on spaces other than $\mathcal{E}_*(\mathbb{R}_+^{d+1})$ and $\mathcal{D}_*(\mathbb{R}^{d+1})$ and we give inversion formulas for these operators.
We consider the operators $\mathcal{K}$ and $\mathcal{K}_W$ defined by :

$$\forall\, x \in \mathbb{R}_+^{d+1},\ \ \mathcal{K}(f)(x) = \mathcal{F}_0{}^{-1}[\xi_\alpha \mathcal{F}_0(f)](x) \tag{36}$$

and

$$\forall\, x \in \mathbb{R}_+^{d+1},\ \ \mathcal{K}_{\mathcal{W}}(f)(x) = \left(\mathcal{F}_W^{\alpha,d}\right)^{-1}[\xi_\alpha \mathcal{F}_W^{\alpha,d}(f)](x), \tag{37}$$

where

$$\forall\, x \in \mathbb{R}_+^{d+1}, \xi_\alpha(x) = \frac{\pi}{2^{2\alpha+1}(\Gamma(\alpha+1))^2} x_{d+1}^{2\alpha+1}. \tag{38}$$

Notations. In what follows, we need the following notations:

- $\mathcal{S}_*^0(\mathbb{R}^{d+1})$ is the subspace of $\mathcal{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that

$$\forall \nu = (\nu_1, ..., \nu_{d+1}) \in \mathbb{N}^{d+1}, \quad D^\nu f(0) = 0,$$

where $D^\nu = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1}...\partial x_{d+1}^{\nu_{d+1}}}$ and $|\nu| = \nu_1 + ... + \nu_{d+1}$.

• $\mathcal{S}_{*,0}(\mathbb{R}^{d+1})$ is the subspace of $\mathcal{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that for all $\nu \in \mathbb{N}^{d+1}$,

$$\int_{\mathbb{R}_+^{d+1}} f(x)x^\nu dx = 0,$$

where for $\nu = (\nu_1, ..., \nu_{d+1}) \in \mathbb{N}^{d+1}$ and $x = (x_1, ..., x_{d+1}) \in \mathbb{R}^{d+1}$, we have $x^\nu = x_1^{\nu_1}...x_{d+1}^{\nu_{d+1}}$.

• $\mathcal{S}^0_{*,0}(\mathbb{R}^{d+1})$ is the subspace of $\mathcal{S}_*(\mathbb{R}^{d+1})$ consisting of functions f such that for all $\nu \in \mathbb{N}^{d+1}$,

$$\int_{\mathbb{R}_+^{d+1}} f(x)m_\nu(x)d\mu_{\alpha,d}(x) = 0,$$

where for $\nu = (\nu_1, ..., \nu_{d+1}) \in \mathbb{N}^{d+1}$ and $x \in \mathbb{R}^{d+1}$, we have

$$m_\nu = \mathcal{R}_W^{\alpha,d}(\frac{u^\nu}{\nu!})(x)$$

and $\nu! = \nu_1!\nu_2!...\nu_{d+1}!$.

Theorem 3. *The transform* ${}^t\mathcal{R}_W^{\alpha,d}$ *is a topological automorphism from* $\mathcal{S}^0_{*,0}(\mathbb{R}^{d+1})$ *onto* $\mathcal{S}_{*,0}(\mathbb{R}^{d+1})$.

Proposition 10.
i) For all f *in* $\mathcal{S}_{*,0}(\mathbb{R}^{d+1})$ *and* g *in* $\mathcal{S}_*(\mathbb{R}^{d+1})$, *the function* $f * g$ *belongs to* $\mathcal{S}_{*,0}(\mathbb{R}^{d+1})$.
ii) For all f *in* $\mathcal{S}^0_{*,0}(\mathbb{R}^{d+1})$ *and* g *in* $\mathcal{S}_*(\mathbb{R}^{d+1})$, *the function* $f *_W g$ *belongs to* $\mathcal{S}^0_{*,0}(\mathbb{R}^{d+1})$.

Proof. We deduce these results from the relation (25), Theorem 1 and the properties of the classical Fourier transform $\mathcal{F}_0$. □

Proposition 11.
i) The operator $\mathcal{K}$, *(respectively* $\mathcal{K}_W$*), is a topological automorphism of* $\mathcal{S}_{*,0}(\mathbb{R}^{d+1})$, *(respectively* $\mathcal{S}^0_{*,0}(\mathbb{R}^{d+1})$*).*
ii) For all f *in* $\mathcal{S}^0_{*,0}(\mathbb{R}^{d+1})$, *we have :*

$$\mathcal{K}_W(f) = \left({}^t\mathcal{R}_W^{\alpha,d}\right)^{-1} \circ \mathcal{K} \circ {}^t\mathcal{R}_W^{\alpha,d}. \tag{39}$$

Proof. i) The mapping $f \to \xi_\alpha f$ is a topological automorphism of $\mathcal{S}^0_*(\mathbb{R}^{d+1})$. We deduce the result from Theorem 1 and the fact that $\mathcal{F}_0$ is a topological automorphism from $\mathcal{S}_{*,0}(\mathbb{R}^{d+1})$ onto $\mathcal{S}^0_*(\mathbb{R}^{d+1})$.
ii) We obtain the result from the relations (37) and (31), and Theorem 3. □

Proposition 12. *i) For all* $f \in \mathcal{S}_{*,0}\left(\mathbb{R}^{d+1}\right)$ *and* $g \in \mathcal{S}_{*}\left(\mathbb{R}^{d+1}\right)$, *we have*

$$\mathcal{K}(f * g) = \mathcal{K}(f) * g. \tag{40}$$

ii) For all $f \in \mathcal{S}^0_{*,0}\left(\mathbb{R}^{d+1}\right)$ *and* $g \in \mathcal{S}_{*}\left(\mathbb{R}^{d+1}\right)$, *we have*

$$\mathcal{K}_W(f *_W g) = \mathcal{K}_W(f) *_W g. \tag{41}$$

Proof. We obtain these relations from (36), (37), Proposition 9, the relation (25), and the properties of the classical transform $\mathcal{F}_0$ and of the classical convolution product on $\mathbb{R}^{d+1}$. □

Theorem 4. *We have the following inversion formulas for the operators* $\mathcal{R}^{\alpha,d}_W$ *and* ${}^t\mathcal{R}^{\alpha,d}_W$.
i) For all $f \in \mathcal{S}^0_{*,0}\left(\mathbb{R}^{d+1}\right)$, *we have*

$$f = \mathcal{R}^{\alpha,d}_W \mathcal{K}\, {}^t\mathcal{R}^{\alpha,d}_W(f). \tag{42}$$

ii) For all $f \in \mathcal{S}_{*,0}\left(\mathbb{R}^{d+1}\right)$, *we have*

$$f = {}^t\mathcal{R}^{\alpha,d}_W \mathcal{K}_W \mathcal{R}^{\alpha,d}_W(f). \tag{43}$$

iii) For all $f \in \mathcal{S}_{*,0}\left(\mathbb{R}^{d+1}\right)$, *we have*

$$f = \mathcal{K}\, {}^t\mathcal{R}^{\alpha,d}_W \mathcal{R}^{\alpha,d}_W(f). \tag{44}$$

iv) For all $f \in \mathcal{S}^0_{*,0}\left(\mathbb{R}^{d+1}\right)$, *we have*

$$f = \mathcal{K}_W \mathcal{R}^{\alpha,d}_W\, {}^t\mathcal{R}^{\alpha,d}_W(f). \tag{45}$$

Proof. i) Let f be in $\mathcal{S}^0_{*,0}\left(\mathbb{R}^{d+1}\right)$. Using Theorem 1, the relations (28) and (25), and the inversion formula for the classical Fourier $\mathcal{F}_0$, we obtain for all $x \in \mathbb{R}^{d+1}_+$,

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}^{d+1}_+} \mathcal{F}^{\alpha,d}_W(f)\,(y)\,\Lambda_{\alpha,d}(-x,y)d\mu_{\alpha,d}(y)\\
&= \int_{\mathbb{R}^{d+1}_+} \mathcal{F}^{\alpha,d}_W(f)\,(y)\,\mathcal{R}^{\alpha,d}_W(e^{i\langle .,y'\rangle}\cos(.y_{d+1}))(x)d\mu_{\alpha,d}(y)\\
&= \mathcal{R}^{\alpha,d}_W\left(\int_{\mathbb{R}^{d+1}_+} \mathcal{F}^{\alpha,d}_W(f)\,(y)\,e^{i\langle .,y'\rangle}\cos(.y_{d+1})d\mu_{\alpha,d}(y)\right)(x)\\
&= \mathcal{R}^{\alpha,d}_W\left(\int_{\mathbb{R}^{d+1}_+} \mathcal{F}_0 o\, {}^t\mathcal{R}^{\alpha,d}_W(f)(y)e^{i\langle .,y'\rangle}\cos(.y_{d+1})d\mu_{\alpha,d}(y)\right)(x)\\
&= \mathcal{R}^{\alpha,d}_W \mathcal{F}_0^{-1}(\xi_\alpha \mathcal{F}_0 o\, {}^t\mathcal{R}^{\alpha,d}_W)(f)(x)\\
&= \mathcal{R}^{\alpha,d}_W \mathcal{K}\, {}^t\mathcal{R}^{\alpha,d}_W(f)(x).
\end{aligned}$$

ii) From the relation (42) and the relation (25), we obtain

$$\forall f \in \mathcal{S}_{*,0}\left(\mathbb{R}^{d+1}\right), f = {}^t\mathcal{R}_W^{\alpha,d}\mathcal{K}_W\mathcal{R}_W^{\alpha,d}(f).$$

iii) We obtain the relation (44) by writing the relation (42) for the function $\mathcal{R}_W^{\alpha,d}(f)$.
iv) We obtain the relation (45) by writing the relation (43) for the function ${}^t\mathcal{R}_W^{\alpha,d}(f)$. □

Corollary 1. *The operator $\mathcal{R}_W^{\alpha,d}$ is a topological isomorphism from $\mathcal{S}_{*,0}\left(\mathbb{R}^{d+1}\right)$ onto $\mathcal{S}_{*,0}^0\left(\mathbb{R}^{d+1}\right)$.*

Proof. We deduce the result from Proposition 10, i), Theorem 3 and the relation (42) □

Corollary 2. *For all f in $\mathcal{S}_{*,0}\left(\mathbb{R}^{d+1}\right)$ and g in $\mathcal{S}\left(\mathbb{R}^{d+1}\right)$, we have*

$$\mathcal{R}_W^{\alpha,d}(f*g) = \mathcal{R}_W^{\alpha,d}(f) *_W \ ({}^t\mathcal{R}_W^{\alpha,d})^{-1}(g). \tag{46}$$

Proof. Using the relations (33), (42), (40) and (44), we obtain

$$\begin{aligned}(\mathcal{R}_W^{\alpha,d})^{-1}[\mathcal{R}_W^{\alpha,d}(f) *_W \ ({}^t\mathcal{R}_W^{\alpha,d})^{-1}(g)] &= \mathcal{K}{}^t\mathcal{R}_W^{\alpha,d}[\mathcal{R}_W^{\alpha,d}(f) *_W \ ({}^t\mathcal{R}_W^{\alpha,d})^{-1}(g)] \\ &= \mathcal{K}[{}^t\mathcal{R}_W^{\alpha,d}\mathcal{R}_W^{\alpha,d}(f) * g] = [\mathcal{K}{}^t\mathcal{R}_W^{\alpha,d}\mathcal{R}_W^{\alpha,d}(f)] * g = f * g\end{aligned}$$

Thus, we obtain the result from Corollary 1. □

5 Weinstein Wavelets and Weinstein Wavelet transform

Definition 5. *A measurable function g on $\mathbb{R}_+^{d+1}$ is a classical wavelet on $\mathbb{R}_+^{d+1}$ if it satisfies, for almost all $x \in \mathbb{R}_+^{d+1}$, the condition :*

$$0 < C_g^0 = \int_0^\infty |\mathcal{F}_0(g)(\lambda x)|^2 \frac{d\lambda}{\lambda} < \infty, \tag{47}$$

where $\mathcal{F}_0$ is the classical Fourier transform given by the relation (31).

Notations. We denote by :
• $L^1(\mathbb{R}^{d+1})$, the space of integrable functions on $\mathbb{R}^{d+1}$ with respect to the Lebesgue measure.
• $L^2(\mathbb{R}^{d+1})$, the space of square integrable functions on $\mathbb{R}^{d+1}$ with respect to the Lebesgue measure.
• $g_{a,x}, a > 0, x \in \mathbb{R}^{d+1}$, the family of classical wavelets on $\mathbb{R}^{d+1}$ in $L^2(\mathbb{R}^{d+1})$ defined by :

$$\forall y \in \mathbb{R}^{d+1}, g_{a,x}(y) = H_a(g)(x-y), \tag{48}$$

where H_a is the dilatation operator given by :

$$\forall x \in \mathbb{R}^{d+1}, H_a(g)(x) = g_a^0(x) = \frac{1}{a^{d+1}} g(\frac{x}{a}). \tag{49}$$

Definition 6. *Let g be a classical Wavelet on $\mathbb{R}^{d+1}$ in $L^2(\mathbb{R}^{d+1})$. The classical continuous Wavelet transform S_g on $\mathbb{R}^{d+1}$ is defined for regular functions f on $\mathbb{R}^{d+1}$ by :*

$$\forall x \in \mathbb{R}^{d+1}, S_g(f)(a,x) = \int_{\mathbb{R}^{d+1}} f(y)\overline{g_{a,x}(y)}dy, \tag{50}$$

where $g_{a,x}, a > 0, x \in \mathbb{R}^{d+1}$ are the family given by the relation (48).

Remark 3. *The transform S_g can also be written in the form :*

$$S_g(f)(a,x) = f * \overline{g_a^0}(x), \tag{51}$$

where $*$ is the clssical convolution product given by the relation (34).

The transform S_g has been introduced in [10]. Various properties of this transform were studied by many anthors (see [10] and [16]). In particular, we have the following results.

Proposition 13. *i)(Plancherel formula) For all $f \in L^2(\mathbb{R}^{d+1})$ we have :*

$$\int_{\mathbb{R}^{d+1}} |f(x)|^2 d\mu_{\alpha,d}(x) = \frac{1}{C_g^0} \int_0^\infty \int_{\mathbb{R}^{d+1}} |S_g(f)(a,x)|^2 dx \frac{da}{a}. \tag{52}$$

ii)(Inversion formula) For all f in $L^1(\mathbb{R}^{d+1})$ such that $\mathcal{F}_0(f)$ belongs to $L^1(\mathbb{R}^{d+1})$, we have

$$f(x) = \frac{1}{C_g^0} \int_0^\infty \left(\int_{\mathbb{R}^{d+1}} S_g(f)(a,y) g_{a,x}(y) dy \right) \frac{da}{a}, \ a.e \ \ x \in \mathbb{R}^{d+1}. \tag{53}$$

Here, the inner integral and the outer integral are absolutely convergent, but possibly not the double integral.

Definition 7. *A Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ is a measurable function g on $\mathbb{R}_+^{d+1}$ satisfying for almost all $x \in \mathbb{R}_+^{d+1}$, the condition :*

$$0 < C_g = \int_0^\infty |\mathcal{F}_W^{\alpha,d}(g)(\lambda x)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{54}$$

Example 4. *The function $g_t, t > 0$, given by :*

$$\forall x \in \mathbb{R}_+^{d+1}, g_t(x) = -\frac{d}{dt}\phi_t(x) \tag{55}$$

where ϕ_t is the function given by the relation (11) is a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ and we have $C_{g_t} = \frac{1}{8t^2}$.

Proposition 14. *A function g is a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $\mathcal{S}\left(\mathbb{R}^{d+1}\right)$, respectively $\mathcal{S}_{*,0}^0\left(\mathbb{R}^{d+1}\right)$ if and only if the function ${}^t\mathcal{R}_W^{\alpha,d}(g)$ is a classical Wavelet on $\mathbb{R}_+^{d+1}$ in $\mathcal{S}\left(\mathbb{R}^{d+1}\right)$, respectively $\mathcal{S}_{*,0}\left(\mathbb{R}^{d+1}\right)$ and we have*

$$C^0_{{}^t\mathcal{R}_W^{\alpha,d}(g)} = C_g. \tag{56}$$

Proof. We deduce these results from Proposition 6, Theorem 3 and the relation (31). □

Let $a \in]0, +\infty[$ and g be a regular function on $\mathbb{R}^{d+1}$. We consider the function g_a given by :

$$\forall x \in \mathbb{R}^{d+1}, g_a(x) = \frac{1}{a^{2\alpha+d+2}} g\left(\frac{x}{a}\right). \tag{57}$$

The function g_a satisfies the following properties.

Proposition 15. *i) For all g in $L^2_\alpha(\mathbb{R}_+^{d+1})$, the function g_a belongs to $L^2_\alpha(\mathbb{R}_+^{d+1})$ and we have*

$$\forall x \in \mathbb{R}_+^{d+1}, \mathcal{F}_W^{\alpha,d}(g_a)(x) = \mathcal{F}_W^{\alpha,d}(g)(ax). \tag{58}$$

ii) For all g in $\mathcal{S}\left(\mathbb{R}^{d+1}\right)$ (respectively $\mathcal{S}_{*,0}^0\left(\mathbb{R}^{d+1}\right)$), the function g_a belongs to $\mathcal{S}\left(\mathbb{R}^{d+1}\right)$ (respectively $\mathcal{S}_{*,0}^0\left(\mathbb{R}^{d+1}\right)$) and we have

$$g_a = ({}^t\mathcal{R}_W^{\alpha,d})^{-1} \circ H_a \circ^t \mathcal{R}_W^{\alpha,d}(g). \tag{59}$$

Definition 8. *Let g be a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $L^2_\alpha(\mathbb{R}_+^{d+1})$. The Weinstein continuous Wavelet transform on $\mathbb{R}_+^{d+1}$ is defined for regular functions f on $\mathbb{R}_+^{d+1}$ by :*

$$S_g^W(f)(a,x) = \int_{\mathbb{R}_+^{d+1}} f(y)\overline{g_{a,x}(y)}d\mu_{\alpha,d}(y), \tag{60}$$

where $g_{a,x}, a > 0, x \in \mathbb{R}_+^{d+1}$ are the family of Weinstein Wavelets on $\mathbb{R}_+^{d+1}$ in $L^2_\alpha(\mathbb{R}_+^{d+1})$ given by :

$$\forall y \in \mathbb{R}_+^{d+1}, g_{a,x}(y) = T_x g_a(y). \tag{61}$$

Here $T_x, x \in \mathbb{R}_+^{d+1}$, are the Weinstein translation operators given by the relation (19).

Remark 4. *The transform S_g^W can also be written in the form*

$$S_g^W(f)(a,x) = f *_W \overline{g_a}(x), \tag{62}$$

where $*_W$ is the Weinstein convolution product given by the relation (23).

Example 5. *We consider the function ϕ_t given by the relation (11). Using the relation (20), we deduce that the family $g_{a,x}, a > 0, x \in \mathbb{R}_+^{d+1}$ given by*

$$\forall y \in \mathbb{R}_+^{d+1}, g_{a,x}(y) = -T_x(\frac{d}{dt}\phi_t)(y) \tag{63}$$

is a family of Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $\mathcal{S}_*\left(\mathbb{R}^{d+1}\right)$.

Theorem 5. *(Plancherel formula for S_g^W)*
For all $f \in L^2_\alpha(\mathbb{R}_+^{d+1})$ we have :

$$\int_{\mathbb{R}_+^{d+1}} |f(x)|^2 \, d\mu_{\alpha,d}(x) = \frac{1}{C_g}\int_0^\infty \int_{\mathbb{R}_+^{d+1}} \left|S_g^W(f)(a,x)\right|^2 d\mu_{\alpha,d}(x)\frac{da}{a}. \tag{64}$$

Proof. Using Fubini-$Tonnelli'$s theorem, Proposition 9, ii) and the relations (62), (58), we get

$$\frac{1}{C_g}\int_0^\infty \int_{\mathbb{R}_+^{d+1}} \left|S_g^W(f)(a,x)\right|^2 d\mu_{\alpha,d}(x)\frac{da}{a} = \frac{1}{C_g}\int_0^\infty (\int_{\mathbb{R}_+^{d+1}} |f *_W \overline{g_a}(x)|^2 \, d\mu_{\alpha,d}(x))\frac{da}{a}$$

$$= \frac{1}{C_g}\int_0^\infty (\int_{\mathbb{R}_+^{d+1}} \left|\mathcal{F}_W^{\alpha,d}(f)(x)\right|^2 \left|\mathcal{F}_W^{\alpha,d}(\overline{g_a})(x)\right|^2 d\mu_{\alpha,d}(x))\frac{da}{a}$$

$$= \int_{\mathbb{R}_+^{d+1}} \left|\mathcal{F}_W^{\alpha,d}(f)(x)\right|^2 (\frac{1}{C_g}\int_0^\infty \left|\mathcal{F}_W^{\alpha,d}(g)(ax)\right|^2 \frac{da}{a})d\mu_{\alpha,d}(x)).$$

On the other hand, using relation (54), we have for almost all $x \in \mathbb{R}_+^{d+1}$

$$\frac{1}{C_g}\int_0^\infty \left|\mathcal{F}_W^{\alpha,d} g(ax)\right|^2 \frac{da}{a} = 1.$$

Then using the relation (18), we obtain

$$\frac{1}{C_g}\int_0^\infty \int_{\mathbb{R}_+^{d+1}} \left|S_g^W(f)(a,x)\right|^2 d\mu_{\alpha,d}(x)\frac{da}{a} = \int_{\mathbb{R}_+^{d+1}} |f(x)|^2 \, d\mu_{\alpha,d}(x).$$

□

The following theorem gives an inversion formula for the transform S_g^W.

Theorem 6. *Let g be a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $L^2_\alpha(\mathbb{R}_+^{d+1})$. For all f in $L^1_\alpha(\mathbb{R}_+^{d+1})$, respectively $L^2_\alpha(\mathbb{R}_+^{d+1})$, such that $\mathcal{F}_W^{\alpha,d}(f)$ belongs to $L^1_\alpha(\mathbb{R}_+^{d+1})$, respectively $L^1_\alpha(\mathbb{R}_+^{d+1}) \bigcap L^2_\alpha(\mathbb{R}_+^{d+1})$, we have*

$$f(x) = \frac{1}{C_g}\int_0^\infty \left(\int_{\mathbb{R}_+^{d+1}} S_g^W(f)(a,y) g_{a,x}(y) d\mu_{\alpha,d}(x)\right)\frac{da}{a},\ a.e, \quad x \in \mathbb{R}_+^{d+1}. \tag{65}$$

The inner integral and the outer integral are absolutely convergent, but possibly not the double integral.

Proof. Using analogous proof as for Theorem 6. III. 3 of [15] page 99, we obtain the relation (65) □

6 Inversion formulas for the Weinstein intertwining operator and its dual using Weinstein wavelets

In this section, we establish relations between the Weinstein continuous wavelet transform S_g^W on $\mathbb{R}_+^{d+1}$ and the classical continuous wavelet transform S_g on $\mathbb{R}_+^{d+1}$. Using the inversion formulas for the transforms S_g^W and S_g, we deduce relations which give the inverse operators of the Weinstein intertwining operator $\mathcal{R}_W^{\alpha,d}$ and its dual ${}^t\mathcal{R}_W^{\alpha,d}$.

Theorem 7. *i) Let g be a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $\mathcal{D}\left(\mathbb{R}^{d+1}\right)$, respectively $\mathcal{S}\left(\mathbb{R}^{d+1}\right)$. Then for all f in the same space as g, we have for all $x \in \mathbb{R}_+^{d+1}$*

$$S_g^W(f)(a,x) = ({}^t\mathcal{R}_W^{\alpha,d})^{-1}[S_{{}^t\mathcal{R}_W^{\alpha,d}(g)}({}^t\mathcal{R}_W^{\alpha,d}(f))(a,.)](x). \tag{66}$$

ii) Let g be a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $\mathcal{S}_0^0\left(\mathbb{R}^{d+1}\right)$. Then for all f in $\mathcal{S}_0\left(\mathbb{R}^{d+1}\right)$, we have

$$\forall x \in \mathbb{R}_+^{d+1}, S_{{}^t\mathcal{R}_W^{\alpha,d}(g)}(f)(a,x) = ({}^t\mathcal{R}_W^{\alpha,d})^{-1}[S_g^W(\mathcal{R}_W^{\alpha,d}(f))(a,.)](x) \tag{67}$$

Proof. We deduce the relations (66) and (67) from the relations (51), (62) and properties of the Weinstein convolution product. □

Theorem 8. *Let g be a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $\mathcal{S}_0^0\left(\mathbb{R}^{d+1}\right)$. Then*
i) For all f in $\mathcal{S}_0^0\left(\mathbb{R}^{d+1}\right)$, we have for all $x \in \mathbb{R}_+^{d+1}$

$$S_g^W(f)(a,x) = a^{-2\alpha-1}\mathcal{R}_W^{\alpha,d}[S_{\mathcal{K}({}^t\mathcal{R}_W^{\alpha,d}(g))}({}^t\mathcal{R}_W^{\alpha,d}(f))(a,.)](x)$$

ii)For all f in $\mathcal{S}_0\left(\mathbb{R}^{d+1}\right)$, we have for all $x \in \mathbb{R}_+^{d+1}$

$$S_{{}^t\mathcal{R}_W^{\alpha,d}(g)}(f)(a,x) = a^{-2\alpha-1}({}^t\mathcal{R}_W^{\alpha,d})[S^W_{\mathcal{K}_W(g)}(\mathcal{R}_W^{\alpha,d}(f))(a,.)](x) \qquad (68)$$

Proof. We obtain these relations from Theorem 7, Theorem 4 and the fact that

$$\mathcal{K}[({}^t\mathcal{R}_W^{\alpha,d}(g))^0_a] = a^{-2\alpha-1}[\mathcal{K}{}^t\mathcal{R}_W^{\alpha,d}(g)]^0_a$$

and

$$\mathcal{K}_W(g_a) = a^{-2\alpha-1}(\mathcal{K}_W(g))_a.$$

□

Theorem 9. *Let g be a Weinstein Wavelet on $\mathbb{R}_+^{d+1}$ in $\mathcal{S}_0^0\left(\mathbb{R}^{d+1}\right)$. Then:*
i) for all f in $\mathcal{S}_0^0\left(\mathbb{R}^{d+1}\right)$, we have for all $x \in \mathbb{R}_+^{d+1}$

$$({}^t\mathcal{R}_W^{\alpha,d})^{-1}(f)(x)$$
$$= \frac{1}{C_g}\int_0^\infty\left(\int_{\mathbb{R}_+^{d+1}}\mathcal{R}_W^{\alpha,d}[S_{\mathcal{K}({}^t\mathcal{R}_W^{\alpha,d}(g))}(f)(a,.)](y)g_{a,x}(y)d\mu_{\alpha,d}(y)\right)\frac{da}{a^{2\alpha+2}};$$

ii) for all f in $\mathcal{S}_0\left(\mathbb{R}^{d+1}\right)$, we have for all $x \in \mathbb{R}_+^{d+1}$

$$(\mathcal{R}_W^{\alpha,d})^{-1}(f)(x)$$
$$= \frac{1}{C^0_{{}^t\mathcal{R}_W^{\alpha,d}(g)}}\int_0^\infty\left(\int_{\mathbb{R}_+^{d+1}}{}^t\mathcal{R}_W^{\alpha,d}[S^W_{\mathcal{K}_W(g)}(f)(a,.)](y){}^t\mathcal{R}_W^{\alpha,d}(g)_{a,x}(y)d\mu_{\alpha,d}(y)\right)\frac{da}{a^{2\alpha+2}}.$$

Proof. We deduce the relations (69) and (69) from Theorem 8, Theorem 3 and the relation (53) □

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