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Graded near-rings

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Abstract

In this paper, we consider graded near-rings over a monoid G as generalizations of graded rings over groups, and study some of their basic properties. We give some examples of graded near-rings having various interesting properties, and we define and study the G^{op}-graded ring associated to a G-graded abelian near-ring, where G is a left cancellative monoid and G^{op} is its opposite monoid. We also compute the graded ring associated to the graded near-ring of polynomials (over a commutative ring R) whose constant term is zero.

Introduction

Near-rings are generalizations of rings: addition is not necessarily abelian and only one distributive law holds. They arise in a natural way in the study of mappings on groups: the set $M(G)$ of all maps of a group $(G,+)$ into itself endowed with pointwise addition and composition of functions is a near-ring. Another classic example of a near-ring is the set $R[X]$ of all polynomials over a commutative ring R with respect to addition and substitution of polynomials.

The concept of a ring graded by a group is well-known in the mathematical literature (see, e.g., [4]). The idea of writing this paper came to us from noticing that some important near-rings, such as the near-ring of polynomials over a commutative ring R or the near-ring of affine maps on a vector space V over a field K, can be naturally graded by a monoid (see Section 2 below).

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Therefore, we were lead to considering graded near-rings over a monoid as generalizations of graded rings over groups.

The paper is organized as follows. In Section 1, we present some basic properties of near-rings graded by a monoid. In Section 2, we give some interesting examples of graded near-rings. In Section 3, we associate to any G-graded abelian near-ring a G^{op}-graded ring, where G is a left cancellative monoid, and we compute the graded ring associated to the graded near-ring of polynomials (over a commutative ring R) whose constant term is zero.

For general background on the theory of near-rings we refer the reader to the monographs written by Pilz [5], Meldrum [3] and Clay [1]. We only briefly recall some basic definitions and notations which will be used throughout the paper.

A *(right) near-ring* is a set N with two binary operations $+$ and $\cdot$ such that:

(1) $(N, +)$ is a group (not necessarily abelian), with the neutral element denoted by 0;

(2) $(N, \cdot)$ is a semigroup;

(3) $(a + b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in N$ ("the right distributive law").

If $(N, \cdot)$ is a monoid, we say that N is a *near-ring with identity.* A *subnear-ring* of a near-ring N is a subgroup M of $(N, +)$ such that $a \cdot b \in M$ for all $a, b \in M$. Any near-ring N has two important subnear-rings: $N_0 = \{n \in N \mid n \cdot 0 = 0\}$, called the *zero symmetric part* of N, and $N_c = \{n \in N \mid n \cdot 0 = n\} = \{n \in N \mid \forall\, a \in N,\, n \cdot a = n\}$, called the *constant part* of N. We say that a near-ring N is *zero symmetric* if $N = N_0$, and *constant* if $N = N_c$. A near-ring N is called *abelian* if the additive group $(N, +)$ is abelian, and *commutative* if the semigroup $(N, \cdot)$ is abelian. If N is a near-ring, then we denote by $N_d = \{d \in N \mid d(r + s) = dr + ds,$ for all $r, s \in N\}$ the set of distributive elements of N. If N is an abelian near-ring, then N_d is a subring of N.

If N and N' are near-rings, then a map $\varphi : N \to N'$ is a *near-ring morphism* in case for all $m, n \in N$ we have $\varphi(m + n) = \varphi(m) + \varphi(n)$ and $\varphi(m \cdot n) = \varphi(m) \cdot \varphi(n)$. A morphism $\varphi : N \to N'$ of near-rings with identity is also required to be unitary, i.e. $\varphi(1_N) = 1_{N'}$.

If N is a near-ring, then a normal subgroup I of $(N, +)$ is called an *ideal* of N if:

(a) $an \in I$, for all $a \in I$ and $n \in N$;

(b) $m(n + a) - mn \in I$, for all $a \in I$ and $m, n \in N$.

Normal subgroups I of $(N,+)$ with (a) are called *right ideals* of N, and normal subgroups I of $(N,+)$ with (b) are called *left ideals* of N.

If N is a near-ring, a group $(\Gamma,+)$ is called an N-*group* (or an N-*near-module*) if there exists an external multiplication $\mu : N\times\Gamma\to\Gamma$, $(n,g)\mapsto ng$ such that for all $g\in\Gamma$ an $m,n\in N$ we have

$$(m+n)g = mg+ng \quad \text{and} \quad (mn)g = m(ng).$$

We usually denote the N-group above as ${}_N\Gamma$. An N-*subgroup* of ${}_N\Gamma$ is a subgroup Δ of Γ with $nh\in\Delta$ for all $n\in N$ and $h\in\Delta$. An *ideal* of ${}_N\Gamma$ is a normal subgroup Δ of Γ such that for all $n\in N$, $g\in\Gamma$, and $\delta\in\Delta$, we have $n(g+\delta)-ng\in\Delta$.

1 Graded near-rings

Unless otherwise stated, G denotes a multiplicatively written monoid with identity element e. If $\{N_\sigma\}_{\sigma\in G}$ is a family of additive normal subgroups of a near-ring N, then we may consider their sum $\sum_{\sigma\in G} N_\sigma$, i.e. the set of all finite sums of elements of different N_σ's. The sum $\sum_{\sigma\in G} N_\sigma$ is called an *internal direct sum* and we write $\bigoplus_{\sigma\in G} N_\sigma$ if each element of $\sum_{\sigma\in G} N_\sigma$ has a unique representation as a finite sum of elements of different N_σ's.

Definition 1.1. We say that a near-ring N is G-*graded* if there exists a family $\{N_\sigma\}_{\sigma\in G}$ of additive normal subgroups of N such that

1) $N=\bigoplus_{\sigma\in G} N_\sigma$ (internal direct sum);

2) $N_\sigma N_\tau\subseteq N_{\sigma\tau}$, for all $\sigma,\tau\in G$.

The set $h(N)=\bigcup_{\sigma\in G} N_\sigma$ is the set of *homogeneous elements* of N. A nonzero element $n\in N_\sigma$ is said to be *homogeneous of degree* σ and we write $\deg(n)=\sigma$. An element $n\in N$ has a unique decomposition as $n=\sum_{\sigma\in G} n_\sigma$, with $n_\sigma\in N_\sigma$ for all $\sigma\in G$, where the sum is finite, i.e. almost all n_σ are zero.

Remark 1.2. Since $N_eN_e\subseteq N_e$, we have that N_e is a subnear-ring of N.

Remark 1.3. Since $N_eN_\sigma\subseteq N_\sigma$ for all $\sigma\in G$, it follows that N_σ is an N_e-subgroup of ${}_{N_e}N$ for all $\sigma\in G$.

Definition 1.4. Let $N=\bigoplus_{\lambda\in G} N_\lambda$ be a G-graded near-ring and $\sigma\in G$.

1) An element $x\in N_\sigma$ is called σ-*distributive* in case for any family $(y_\tau)_{\tau\in G}$ of finite support of homogeneous elements in N (with $y_\tau\in N_\tau$, for all $\tau\in G$),

the following distributivity condition is satisfied:

$$x\left(\sum_{\tau\in G} y_\tau\right) = \sum_{\tau\in G} xy_\tau.$$

2) The G-graded near-ring N is called *σ-distributive* if any homogeneous element in N of degree σ is σ-distributive.

Proposition 1.5. *If $x \in N_\sigma$ is σ-distributive for any $\sigma \in G$, then x is distributive (i.e. $x \in N_d$).*

Proof. Let $y = \sum_{\tau\in G} y_\tau$ and $z = \sum_{\tau\in G} z_\tau$ be two arbitrary elements in N (with $y_\tau, z_\tau \in N_\tau$, for all $\tau \in G$). If $x \in N_\sigma$ is σ-distributive for any $\sigma \in G$, then we may write

$$\begin{aligned} x(y+z) &= x\left(\sum_{\tau\in G}(y_\tau + z_\tau)\right) = \sum_{\tau\in G} x(y_\tau + z_\tau) \\ &= \sum_{\tau\in G} xy_\tau + \sum_{\tau\in G} xz_\tau = x\left(\sum_{\tau\in G} y_\tau\right) + x\left(\sum_{\tau\in G} z_\tau\right) \\ &= xy + xz. \end{aligned}$$

Hence $x \in N_d$. □

If X is a nontrivial additive subgroup of N, then we write $X_\sigma = X \cap N_\sigma$ for $\sigma \in G$. We say that X is *G-graded* in case $X = \sum_{\sigma\in G} X_\sigma$. In particular, when X is a subnear-ring, a left ideal, a right ideal, an ideal, respectively, we obtain the notions of G-graded subnear-ring, G-graded left ideal, G-graded right ideal, G-graded ideal, respectively. If I is a graded ideal of N, then the factor near-ring N/I is a G-graded near-ring $N/I = \bigoplus_{\sigma\in G}(N/I)_\sigma$ with gradation defined by $(N/I)_\sigma = N_\sigma + I/I$, for all $\sigma \in G$.

Remark 1.6. Let $N = \bigoplus_{\sigma\in G} N_\sigma$ be a G-graded near-ring and I be a normal subgroup of $(N, +)$. Then it is easy to see that the following assertions hold:

(a) I is a graded right ideal of N if and only if $an_\sigma \in I$, for all $a \in I$, $n_\sigma \in N_\sigma$, and $\sigma \in G$;

(b) I is a graded left ideal of N if and only if $m_\sigma(n + a) - m_\sigma n \in I$, for all $a \in I$, $n \in N$, $m_\sigma \in N_\sigma$, and $\sigma \in G$.

(c) I is a graded ideal of N if and only if conditions (a) and (b) from above are satisfied.

Remark 1.7. Let G be a group and N be a ring. Clearly, if N is a G-graded near-ring, then N is simply a G-graded ring (see [4]).

Proposition 1.8. *If N is a G-graded near-ring, then N_0 is a G-graded subnear-ring of N.*

Proof. Let $n \in N_0$, $n = \sum_{\sigma \in G} n_\sigma$ with $n_\sigma \in N_\sigma$ for all $\sigma \in G$. Since $n0 = 0$, we have $\sum_{\sigma \in G} n_\sigma 0 = 0$, so $n_\sigma 0 = 0$ for all $\sigma \in G$. Hence $n_\sigma \in N_0$ for all $\sigma \in G$. Therefore, $n_\sigma \in N_0 \cap N_\sigma = (N_0)_\sigma$ for all $\sigma \in G$. Clearly, $N_0 = \sum_{\sigma \in G} (N_0)_\sigma$, so N_0 is a G-graded subnear-ring of N. $\square$

Proposition 1.9. *If G is a nontrivial left cancellative monoid and N is a G-graded near-ring, then $N_c = 0$.*

Proof. Let $n \in N_c$, $n = \sum_{\sigma \in G} n_\sigma$ with $n_\sigma \in N_\sigma$ for all $\sigma \in G$. Since $n0 = n$, we have $\sum_{\sigma \in G} n_\sigma 0 = \sum_\sigma n_\sigma$, and thus $n_\sigma 0 = n_\sigma$ for all $\sigma \in G$. Hence $n_\sigma \in N_c$ for all $\sigma \in G$.

Let $\sigma, \tau \in G$, $\sigma \neq \tau$. Since $0 \in N_\tau$, it follows that $n_\sigma 0 \in N_{\sigma\tau}$, so $n_\sigma 0 = n_{\sigma\tau}$. If $\tau' \in G$, $\tau' \neq \tau$, then $n_\sigma 0 = n_{\sigma\tau} = n_{\sigma\tau'} \in N_{\sigma\tau} \cap N_{\sigma\tau'} = 0$, so $n_\sigma 0 = 0$. But $n_\sigma 0 = n_\sigma$, hence $n_\sigma = 0$ for all $\sigma \in G$. Therefore, $n = 0$, and thus $N_c = 0$. $\square$

Remark 1.10. It is easy to see that any ring R may be viewed as a G-graded ring, for any monoid G, by considering the so-called trivial grading on R, i.e. $R_e = R$ and $R_\sigma = 0$ for all $\sigma \neq e$ in G. For near-rings, this is not necessarily true. Indeed, if $(N, +, *)$ is the near-ring with multiplication defined by $a * b = a$, for all $a, b \in N$ (see [5, p. 8]), then $N = N_c$ and from Proposition 1.9 it follows that there is no nontrivial grading on N by any nontrivial monoid G.

Proposition 1.11. *Let G be a nontrivial left cancellative monoid and $N = \bigoplus_{\sigma \in G} N_\sigma$ be a G-graded near-ring with identity 1. If every homogeneous component of degree σ of 1 is σ-distributive, then $1 \in N_e$.*

Proof. Let $1 = \sum_{\sigma \in G} n_\sigma$ be the decomposition of 1 with $n_\sigma \in N_\sigma$. Then for any $a_\lambda \in N_\lambda$ ($\lambda \in G$), we have that $a_\lambda = 1 \cdot a_\lambda = \sum_{\sigma \in G} n_\sigma a_\lambda$ and $n_\sigma a_\lambda \in N_{\sigma\lambda}$. For $\sigma \neq e$, we have $n_\sigma a_\lambda = 0$, and for $\sigma = e$, we have $n_e a_\lambda = a_\lambda$. Therefore, if $\sigma \neq e$, we have $n_\sigma(\sum_{\lambda \in G} a_\lambda) = \sum_{\lambda \in G} n_\sigma a_\lambda = 0$. Hence, if $a = \sum_{\lambda \in G} a_\lambda$, it follows that $n_\sigma a = 0$ for $\sigma \neq e$. For $a = 1$, we obtain that $n_\sigma = 0$ for all $\sigma \neq e$. Hence $1 = n_e \in N_e$. $\square$

Proposition 1.12. *Let G be a nontrivial left cancellative monoid and $N = \bigoplus_{\sigma \in G} N_\sigma$ be a G-graded near-ring with identity 1. Then the homogeneous component of degree e of 1 is an idempotent in N_e.*

Proof. Let $1 = \sum_{\sigma \in G} n_\sigma$ be the decomposition of 1 with $n_\sigma \in N_\sigma$. Then $n_e = 1 \cdot n_e = \sum_{\sigma \in G} n_\sigma n_e$ with $n_\sigma n_e \in N_\sigma$ for all $\sigma \in G$. Then $n_\sigma n_e = 0$, for all $\sigma \neq e$, and $n_e = n_e^2$. Thus n_e is an idempotent in N_e. □

Proposition 1.13. *Let $N = \bigoplus_{\sigma \in G} N_\sigma$ be a G-graded abelian near-ring with identity 1. If ${}_{N_e}N_\sigma$ is an ideal in ${}_{N_e}N$ for all $\sigma \in G$, then N is e-distributive.*

Proof. Let $a_e \in N_e$, $x_\sigma \in N_\sigma$ and $y_\tau \in N_\tau$ be arbitrary homogeneous elements, where $\sigma, \tau \in G$ with $\sigma \neq \tau$. Since ${}_{N_e}N_\sigma$ is an ideal in ${}_{N_e}N$, we have $a_e(x_\sigma + n) - a_e n \in N_\sigma$ for all $n \in N$. In particular, for $n = y_\tau$, we obtain $a_e(x_\sigma + y_\tau) - a_e y_\tau \in N_\sigma$. Since N_σ is an N_e-subgroup of ${}_{N_e}N$ (Remark 1.3), we also have $-a_e x_\sigma \in N_\sigma$. Therefore,

$$a_e(x_\sigma + y_\tau) - a_e y_\tau - a_e x_\sigma \in N_\sigma. \tag{1}$$

Since ${}_{N_e}N_\tau$ is an ideal of ${}_{N_e}N$ and an N_e-subgroup of ${}_{N_e}N$, we similarly obtain

$$a_e(x_\sigma + y_\tau) - a_e y_\tau - a_e x_\sigma \in N_\tau. \tag{2}$$

Hence, from (1) and (2) it follows that $a_e(x_\sigma + y_\tau) - a_e y_\tau - a_e x_\sigma \in N_\sigma \cap N_\tau = \{0\}$, so

$$a_e(x_\sigma + y_\tau) = a_e x_\sigma + a_e y_\tau,$$

for all $a_e \in N_e$, $x_\sigma \in N_\sigma$ and $y_\tau \in N_\tau$. Therefore, any homogeneous element of degree e of N is e-distributive, so N is e-distributive. □

Proposition 1.14. *Let $N = \bigoplus_{\sigma \in G} N_\sigma$ be a G-graded abelian near-ring. Then N is σ-distributive for all $\sigma \in G$ if and only if N is a G-graded ring.*

Proof. ($\Leftarrow$). This is clear.

($\Rightarrow$). Let $n = \sum_{\lambda \in G} \in N$, with $n_\lambda \in N_\lambda$ for all $\lambda \in G$. It is enough to prove that $n(x_\sigma + y_\tau) = nx_\sigma + ny_\tau$, for any homogeneous elements $x_\sigma, y_\tau \in N$ (with $\sigma, \tau \in G$). Since N is σ-distributive for any $\sigma \in G$, from Proposition 1.5 it follows that

$$\begin{aligned} n(x_\sigma + y_\tau) &= \left(\sum_{\lambda \in G} n_\lambda\right)(x_\sigma + y_\tau) = \sum_{\lambda \in G} n_\lambda (x_\sigma + y_\tau) \\ &= \sum_{\lambda \in G} (n_\lambda x_\sigma + n_\lambda y_\tau) = \sum_{\lambda \in G} n_\lambda x_\sigma + \sum_{\lambda \in G} n_\lambda y_\tau \\ &= \left(\sum_{\lambda \in G} n_\lambda\right) x_\sigma + \left(\sum_{\lambda \in G} n_\lambda\right) y_\tau \\ &= nx_\sigma + ny_\tau. \end{aligned}$$

Hence, any $n \in N$ is distributive, so N is a G-graded ring. □

Theorem 1.15. *Let G be a finite group isomorphic to $\mathbb{Z}_2$ and $N = \bigoplus_{\sigma\in G} N_\sigma$ be a G-graded abelian near-ring with identity 1. If ${}_{N_e}N_\sigma$ is an ideal in ${}_{N_e}N$ for all $\sigma \in G$, then N_d is a G-graded subring of N.*

Proof. Let $a = \sum_{\sigma\in G} a_\sigma \in N_d$, with $a_\sigma \in N_\sigma$ for all $\sigma \in G$. We only have to prove that $a_\sigma \in (N_d)_\sigma = N_d \cap N_\sigma$ for all $\sigma \in G$, that is $a_\sigma(x+y) = a_\sigma x + a_\sigma y$, for all $x, y \in N$. It is enough to show that this equality holds when x and y are two homogeneous elements of N, say $x = x_\lambda$ and $y = y_\mu$, with $\lambda, \mu \in G$. Since $a = \sum_{\sigma\in G} a_\sigma \in N_d$, for any homogeneous elements $x_\lambda, y_\mu \in N$ we have

$$a(x_\lambda + y_\mu) = ax_\lambda + ay_\mu.$$

On the left-hand side we have

$$a(x_\lambda + y_\mu) = \left(\sum_{\sigma\in G} a_\sigma\right)(x_\lambda + y_\mu) = \sum_{\sigma\in G} a_\sigma(x_\lambda + y_\mu).$$

On the right-hand side we have

$$\begin{aligned} ax_\lambda + ay_\mu &= \left(\sum_{\sigma\in G} a_\sigma\right)x_\lambda + \left(\sum_{\sigma\in G} a_\sigma\right)y_\mu \\ &= \sum_{\sigma\in G} a_\sigma x_\lambda + \sum_{\sigma\in G} a_\sigma y_\mu \\ &= \sum_{\sigma\in G} (a_\sigma x_\lambda + a_\sigma y_\mu). \end{aligned}$$

Hence, for any homogeneous elements $x_\lambda, y_\mu \in N$ we obtain

$$\sum_{\sigma\in G} a_\sigma(x_\lambda + y_\mu) = \sum_{\sigma\in G} (a_\sigma x_\lambda + a_\sigma y_\mu). \tag{3}$$

Let $G = \{e, \tau\}$, where $\tau \neq e$. From Proposition 1.13, we have

$$a_e(x_\lambda + y_\mu) = a_e x_\lambda + a_e y_\mu. \tag{4}$$

From (3) and (4), it follows that

$$a_\tau(x_\lambda + y_\mu) = a_\tau x_\lambda + a_\tau y_\mu.$$

Hence $a_\sigma(x_\lambda + y_\mu) = a_\sigma x_\lambda + a_\sigma y_\mu$, for all $x_\lambda, y_\mu \in N$, which ends the proof. □

We end this section with some considerations about the category of graded near-rings. Let $\mathcal{N}$ be the category of near-rings. If G is a monoid, we denote by $G - \mathcal{N}$ the category of G-graded near-rings, in which the objects are the G-graded near-rings and the morphisms are the near-ring morphisms $\varphi : N \to N'$ between G-graded near-rings N and N' such that $\varphi(N_\sigma) \subseteq N'_\sigma$. Clearly, for $G = \{e\}$, we have $G - \mathcal{N} = \mathcal{N}$. Note that $\mathcal{N}$ contains, as a full subcategory, the category of rings **Ring**, and $G - \mathcal{N}$ contains, as a full subcategory, the category of G-graded rings $G - \mathbf{Ring}$.

Proposition 1.16. *The category $G - \mathcal{N}$ has arbitrary direct products.*

Proof. Let $(N_i)_{i\in I}$ be a family of G-graded near-rings, where $N_i = \bigoplus_{\sigma\in G}(N_i)_\sigma$, for all $i \in I$. For every $\sigma \in G$, we consider the direct product $\prod_{i\in I}(N_i)_\sigma$ of additive subgroups $(N_i)_\sigma$ of N_i $(i \in I)$. Then

$$N = \bigoplus_{\sigma\in G}\left(\prod_{i\in I}(N_i)_\sigma\right)$$

is a G-graded near-ring, which is the direct product of the family $(N_i)_{i\in I}$ in the category $G - \mathcal{N}$. □

We denote the G-graded near-ring N above by $\prod^{gr}_{i\in I} N_i$ and call it the *direct product of the family of G-graded near-rings* $(N_i)_{i\in I}$. Note that if G is finite or I is a finite set, then $\prod^{gr}_{i\in I} N_i = \prod_{i\in I} N_i$.

2 Examples of graded near-rings

In this section, we give some examples of graded near-rings having various interesting properties.

Example 2.1. Let R be a commutative ring with identity and $R[X]$ be the set of all polynomials in one indeterminate X with coefficients in R. Then $R[X]$ is a zero symmetric near-ring with identity X under addition "+" and substitution "∘" of polynomials, i.e. $f \circ g = f(g(X))$ for all $f, g \in R[X]$ (see [5]). We denote by $R_0[X]$ the set of all polynomials over R whose constant term is zero. $R_0[X]$ is a subnear-ring of $(R[X], +, \circ)$ and $R_0[X] = (R[X])_0$, the zero-symmetric part of $(R[X], +, \circ)$ (see [5, Chap. 7]). If $\mathbb{N}^*$ is the multiplicative monoid of nonzero natural numbers, then $(R_0[X], +, \circ)$ is an $\mathbb{N}^*$-graded near-ring with the grading defined by $(R_0[X])_n = RX^n$, for all $n \in \mathbb{N}^*$. In particular, the degree 1 component is RX. We clearly have $R_0[X] = \bigoplus_{n\in\mathbb{N}^*}(R_0[X])_n$. Since $RX^n \circ RX^m = RX^{nm}$ for all $n, m \in \mathbb{N}^*$, then $(R_0[X])_n \circ (R_0[X])_m \subseteq (R_0[X])_{nm}$ for all $n, m \in \mathbb{N}^*$. Moreover, for all $f \in R_0[X]$, $bX^n \in RX^n$, and $aX \in RX$, we have

$$aX \circ (f(X) + bX^n) - aX \circ f(X) = a(f(X) + bX^n) - af(X) = abX^n \in RX^n,$$

hence every ${}_{RX}RX^n$ is an ideal of ${}_{RX}R_0[X]$.

Remark 2.2. Note that the direct sum decomposition $R[X] = \bigoplus_{n\geq 0} RX^n$ does *not* define an $\mathbb{N}^*$-grading on the near-ring of polynomials $(R[X], +, \circ)$ if we consider $R[X]_0 = R$ and $R[X]_n = RX^n$ for all $n \geq 1$, because $R[X]_0 \circ R[X]_n = R \circ RX^n \subseteq R = R[X]_0$ for all $n \in \mathbb{N}^*$.

Example 2.3. Let V be a finitely dimensional vector space over a field K. Recall that a map $f : V \to V$ is *affine* if it is the sum of a linear map and a constant map: $f = u + a$, where $u \in \mathrm{End}_K(V)$ and $a \in V$ (We identify the constant maps on V with the elements of V). The set $M_{aff}(V)$ of all affine maps on V is a zero symmetric near-ring under pointwise addition of functions and composition of functions (see [5, p. 9]).

Let $G_2 = \{0, 1\}$ be a set with two elements endowed with an additive operation defined by

$$0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 0 = 1, \text{ and } 1 + 1 = 1.$$

It is easy to see that $(G_2, +)$ is a commutative monoid with identity 0 and that G_2 is not left (or right) cancellative. We define a G_2-grading on the near-ring $N = M_{aff}(V)$ as follows:

$$N = N_0 \oplus N_1, \text{ where } N_0 = \mathrm{End}_K(V) \text{ and } N_1 = V.$$

Clearly, N_0 and N_1 are additive subgroups of N and $N = N_0 + N_1$. Moreover, if $f \in N_0 \cap N_1$, then $f = u \in \mathrm{End}_K(V)$ and $f = a \in V$, so $u = a$, and thus $u(x) = a$ for all $x \in V$. Hence $u(0) = a$ and, since $u(0) = 0$, we obtain $a = 0$, which implies $u = 0$. Therefore, $f = 0$, so $N_0 \cap N_1 = 0$, and thus we have the direct sum decomposition of additive groups $M_{aff}(V) = \mathrm{End}_K(V) \oplus V$, that is $N = N_0 \oplus N_1$. Let us check now that $N_\sigma \circ N_\tau \subseteq N_{\sigma+\tau}$, for all $\sigma, \tau \in G_2$. Indeed:

- If $u, v \in \mathrm{End}_K(V)$, then it is clear that $u \circ v \in \mathrm{End}(V)$, so $N_0 \circ N_0 \subseteq N_0$.
- If $u \in \mathrm{End}_K(V)$ and $a \in V$, then $(u \circ a)(x) = u(a) \in V$, for all $x \in V$, so $u \circ a = u(a) \in V$, and thus $N_0 \circ N_1 \subseteq N_1$.
- If $a \in V$ and $u \in \mathrm{End}_K(V)$, then $(a \circ u)(x) = a \circ u(x) = a \in V$, for all $x \in V$, so $a \circ u = a \in V$, hence $N_1 \circ N_0 \subseteq N_1$.
- If $a, b \in V$, then $a \circ b = a \in V$, so $N_1 \circ N_1 \subseteq N_1$.

Therefore, $N = M_{aff}(V)$ is a G_2-graded near-ring. Moreover, $N = N_0 \oplus N_1$ has the following properties:

(i) $N_0 = \mathrm{End}_K(V)$ is a subring of N and $N_0 = N_d$, the distributive part of N.

(ii) $N_1 = V$ is a subnear-ring of N and $N_1 = N_c$, the constant part of N.

For (i), just use the known fact that $N_d = \mathrm{End}_K(V)$ (see [5, Examples 1.12]). Let us prove (ii). Since $a \circ 0 = a$, for all $a \in V$, we have $V \subseteq N_c$. Conversely, if $f = u + a \in N = M_{aff}(V)$ with $u \in \mathrm{End}_K(V)$ and $a \in V$, then

$$\begin{aligned} f \in N_c \quad &\Rightarrow \quad f \circ 0 = f \quad \Rightarrow \quad (u+a) \circ 0 = u + a \\ &\Rightarrow \quad u \circ 0 + a \circ 0 = u + a \\ &\Rightarrow \quad 0 + a = u + a \\ &\Rightarrow \quad u = 0 \quad \Rightarrow \quad f = a \in V, \end{aligned}$$

so $N_c \subseteq V$. Hence $N_c = V$, and so $N_1 = N_c$.

Example 2.4. Let R be a commutative ring with identity and $(R[X], +, \circ)$ be the near-ring of polynomials over R. Let $(R_0[X], +, \circ)$ be the near-ring of polynomials over R whose constant term is zero (see Example 2.1). Let $(G_2 = \{0, 1\}, +)$ be the additive monoid from Example 2.3. We define a G_2-grading on the near-ring $N = R[X]$ by

$$N = N_0 \oplus N_1, \text{ where } N_0 = R_0[X] \text{ and } N_1 = R.$$

We clearly have the direct sum decomposition of additive subgroups

$$R[X] = R_0[X] \oplus R.$$

We check now that $N_\sigma \circ N_\tau \subseteq N_{\sigma+\tau}$, for all $\sigma, \tau \in G_2$:

- If $f = a_n X^n + \cdots + a_1 X, g = b_m X^m + \cdots + b_1 X \in R_0[X]$, then

  $$\begin{aligned} f \circ g &= (a_n X^n + \cdots + a_1 X) \circ (b_m X^m + \cdots + b_1 X) \\ &= a_n (b_m X^m + \cdots + b_1 X)^n + \cdots + a_1 (b_m X^m + \cdots + b_1 X) \in R_0[X], \end{aligned}$$

  so $N_0 \circ N_0 \subseteq N_0$.

- If $f = a_n X^n + \cdots + a_1 X \in R_0[X]$ and $r \in R$, then

  $$(a_n X^n + \cdots + a_1 X) \circ r = a_n r^n + \cdots + a_1 r \in R,$$

  so $f \circ r \in R$, and thus $N_0 \circ N_1 \subseteq N_1$.

- If $r \in R$ and $f \in R_0[X]$, then $r \circ f = r \in R$, hence $N_1 \circ N_0 \subseteq N_1$.

- If $r, s \in R$, then $r \circ s = r \in R$, so $N_1 \circ N_1 \subseteq N_1$.

Therefore, $N = R[X]$ is a G_2-graded near-ring. Moreover, $N = N_0 \oplus N_1$ has the following properties:

(i) $N_0 = R_0[X]$ is a subnear-ring of N and $(N_0)_d$ is a ring containing $(R[X])_d$ as a subring.

(ii) $N_1 = R$ is a subnear-ring of N and $N_1 = N_c$, the constant part of N.

For the first part of (i), see [5, Chapter 7-78]); the second part of (i) is [2, Proposition 1.1(ii)]. In [2], one can also find a description of the distributive elements of the near-rings of polynomials over a commutative ring with identity.

We now prove (ii). Since $r \circ 0 = r$, for all $r \in R$, we have $R \subseteq N_c$. Conversely, if $f = a_nX^n + \cdots + a_1X \in N = R[X]$, then

$$\begin{aligned} f \in N_c \quad &\Rightarrow \quad f \circ 0 = f \\ &\Rightarrow \quad (a_nX^n + \cdots + a_1X + a_0) \circ 0 = a_nX^n + \cdots + a_1X + a_0 \\ &\Rightarrow \quad a_0 = a_nX^n + \cdots + a_1X + a_0 \\ &\Rightarrow \quad n = 0 \quad \Rightarrow \quad f = a_0 \in R, \end{aligned}$$

so $N_c \subseteq R$. Hence $N_c = R$, and so $N_1 = N_c$.

Remark 2.5. As Examples 2.3 and 2.4 show, the condition that the nontrivial monoid G is left cancellative is essential in Proposition 1.9. Indeed, both aforementioned examples are of near-rings graded by a nontrivial monoid which is not left (or right) cancellative, and both near-rings have nonzero constant part.

Example 2.6. Let $(\mathbb{Z}_2, +)$ be the additive abelian group with two elements, i.e. $\mathbb{Z}_2 = \{0, 1\}$ with addition defined by $0 + 0 = 0$, $0 + 1 = 1 + 0 = 1$, and $1 + 1 = 0$. We shall construct a $\mathbb{Z}_2$-graded near-ring as follows. Let H_0 and H_1 be two nonzero abelian groups, and $G = H_0 \times H_1$ be their direct product, which is also an abelian group. Let $M(G) = \{f : G \to G\}$ be the near-ring of all maps from G to G with pointwise addition and composition of functions (see [5, p. 8]). We consider the sets

$$N_0 = \{f : G \to G \mid f(x_0, x_1) \in H_0 \times 0\}$$

and

$$\begin{aligned} N_1 = \{f : G \to G \mid &f(x_0, 0) \in 0 \times H_1,\ f(0, x_1) = (0, 0), \\ &\text{and } f(x_0, x_1) = f(x_0, 0) + f(0, x_1)\}, \end{aligned}$$

which are, clearly, additive subgroups of $(M(G), +)$. Then the sum $N_0 + N_1$ is direct. Indeed, if $f \in N_0 \cap N_1$, then, for all $x_0 \in H_0$ and $x_1 \in H_1$, we have:

$$\begin{gathered} f(x_0, x_1) \in H_0 \times 0, \quad f(x_0, 0) \in 0 \times H_1, \quad f(0, x_1) = (0, 0), \\ f(x_0, x_1) = f(x_0, 0) + f(0, x_1). \end{gathered}$$

From $f(x_0, x_1) \in H_0 \times 0$, it follows that $f(x_0, 0) \in H_0 \times 0$. But we also have $f(x_0, 0) \in 0 \times H_1$, so we obtain $f(x_0, 0) \in (H_0 \times 0) \cap (0 \times H_1) = 0 \times 0$, so $f(x_0, 0) = (0, 0)$. Therefore,

$$f(x_0, x_1) = f(x_0, 0) + f(0, x_1) = (0, 0), \text{ for all } (x_0, x_1) \in G,$$

and thus $f = 0$. Hence $N_0 \cap N_1 = 0$.

Let $N = N_0 \oplus N_1$. Clearly, $(N, +)$ is a subgroup of the abelian group $(M(G), +)$. For any $f + g, f' + g' \in N_0 \oplus N_1$, we may write

$$(f + g) \circ (f' + g') = f \circ (f' + g') + g \circ (f' + g'). \tag{5}$$

For all $(x_0, x_1) \in G = H_0 \times H_1$, we have:

$$\begin{aligned}\big(f \circ (f' + g')\big)(x_0, x_1) &= f\big((f' + g')(x_0, x_1)\big)\\ &= f\big(f'(x_0, x_1) + g'(x_0, x_1)\big)\\ &= f\big((y_0, 0) + (0, y_1)\big)\\ &= f(y_0, y_1) \in H_0 \times 0,\end{aligned}$$

where $f'(x_0, x_1) = (y_0, 0) \in H_0 \times 0$ and $g'(x_0, x_1) = g'(x_0, 0) + g'(0, x_1) = (0, y_1) + (0, 0) = (0, y_1) \in 0 \times H_1$. Therefore,

$$f \circ (f' + g') = h \in N_0, \tag{6}$$

where $h : G \to G$ is defined by $h(x_0, x_1) = f(y_0, y_1)$, for all $(x_0, x_1) \in G$.

Similarly, for all $(x_0, x_1) \in G = H_0 \times H_1$, we also have:

$$\begin{aligned}\big(g \circ (f' + g')\big)(x_0, x_1) &= g\big((f' + g')(x_0, x_1)\big)\\ &= g\big(f'(x_0, x_1) + g'(x_0, x_1)\big)\\ &= g\big((y_0, 0) + (0, y_1)\big)\\ &= g(y_0, y_1) \in 0 \times H_1,\end{aligned}$$

where $f'(x_0, x_1) = (y_0, 0) \in H_0 \times 0$ and $g'(x_0, x_1) = (0, y_1) \in 0 \times H_1$. Therefore,

$$g \circ (f' + g') = k \in N_1, \tag{7}$$

where $k : G \to G$ is defined by $k(x_0, x_1) = g(y_0, y_1)$, for all $(x_0, x_1) \in G$.

Hence, from (5), (6) and (7), it follows that $(f + g) \circ (f' + g') \in N_0 \oplus N_1$, for all $f + g, f' + g' \in N_0 \oplus N_1$, which shows that $N = N_0 \oplus N_1$ is a subnear-ring of $M(G)$.

We prove now that $N_\sigma \circ N_\tau \subseteq N_{\sigma+\tau}$, for all $\sigma, \tau \in \mathbb{Z}_2$:

- If $f, f' \in N_0$, then, for all $(x_0, x_1) \in G$, we have
$$f\big(f'(x_0, x_1)\big) = f(y_0, 0) \in H_0 \times 0,$$
where $f'(x_0, x_1) = (y_0, 0) \in H_0 \times 0$. Hence, $f \circ f' \in N_0$, and so $N_0 \circ N_0 \subseteq N_0$.
- If $f \in N_0$ and $g \in N_1$, then, for all $(x_0, x_1) \in G$, we have
$$\begin{aligned} f\big(g(x_0, x_1)\big) &= f\big(g(x_0, 0) + g(0, x_1)\big) \\ &= f\big((0, y_1) + (0, 0)\big) = f(0, y_1) = (0, 0), \end{aligned}$$
where $g(x_0, 0) = (0, y_1) \in 0 \times H_1$. Hence, $f \circ g = 0$, and so $N_0 \circ N_1 = 0 \subseteq N_1$.
- If $g \in N_1$ and $f \in N_0$, then, for all $(x_0, x_1) \in G$, we have
$$g\big(f(x_0, x_1)\big) = g(y_0, 0) \in 0 \times H_1,$$
where $f(x_0, x_1) = (y_0, 0) \in H_0 \times 0$. Thus $g \circ f \in N_1$, and so $N_1 \circ N_0 \subseteq N_1$.
- If $g, g' \in N_1$, then, for all $(x_0, x_1) \in G$, we have
$$\begin{aligned} g\big(g'(x_0, x_1)\big) &= g\big(g'(x_0, 0) + g'(0, x_1)\big) \\ &= g\big((0, y_1) + (0, 0)\big) = g(0, y_1) = (0, 0), \end{aligned}$$
where $g'(x_0, 0) = (0, y_1) \in 0 \times H_1$. Hence, $g \circ g' = 0$, and so $N_1 \circ N_1 = 0 \subseteq N_0$.

Therefore, $N = N_0 \oplus N_1$ is a $\mathbb{Z}_2$-graded near-ring.

Moreover, N_0 is a subnear-ring of N which is not a ring, because the left distributivity law does not hold: $f \circ (f' + f'') \neq f \circ f' + f \circ f''$, for $f, f', f'' \in N_0$. Indeed, if $(x_0, x_1) \in G$ is arbitrary, and if $f'(x_0, x_1) = (y_0', 0) \in H_0 \times 0$ and $f''(x_0, x_1) = (y_0'', 0) \in H_0 \times 0$, then we may write:
$$\begin{aligned} \big(f \circ (f' + f'')\big)(x_0, x_1) &= f\big((f' + f'')(x_0, x_1)\big) \\ &= f\big(f'(x_0, x_1) + f''(x_0, x_1)\big) \\ &= f\big((y_0', 0) + (y_0'', 0)\big) \end{aligned}$$
and
$$\begin{aligned} (f \circ f' + f \circ f'')(x_0, x_1) &= f\big(f'(x_0, x_1)\big) + f\big(f''(x_0, x_1)\big) \\ &= f(y_0', 0) + f(y_0'', 0). \end{aligned}$$
Since f is not an endomorphism, we have $f\big((y_0', 0) + (y_0'', 0)\big) \neq f(y_0', 0) + f(y_0'', 0)$.

Hence $N = N_0 \oplus N_1$ is a $\mathbb{Z}_2$-graded near-ring which is not a $\mathbb{Z}_2$-graded ring.

3 The graded ring associated to a graded near-ring

Let N be an abelian near-ring and $P = \{\rho_n \mid n \in N\}$ be the subset of the ring $\mathrm{End}(N)$ of endomorphisms of $(N, +)$, consisting of all right multiplication maps on N, that is $\rho_n : N \to N$, $\rho_n(a) = an$, for all $a \in N$. We clearly have that $\rho_n \circ \rho_m = \rho_{mn}$, for all $n, m \in N$, hence P is closed under map composition.

Let $A(N)$ be the associated ring of N, i.e. the subring of $\mathrm{End}(N)$ generated by P (see [6]). Any element of $A(N)$ is a finite sum of right multiplication maps from P. If N has identity 1, then $\rho_n \circ \rho_1 = \rho_1 \circ \rho_n = \rho_n$, for all $n \in N$, and so ρ_1 is the identity of the ring $A(N)$.

Let G be a left cancellative monoid and $N = \bigoplus_{\sigma \in G} N_\sigma$ be a G-graded abelian near-ring. For any $\sigma \in G$, let

$$A(N)_\sigma = \left\{ \sum_{\text{finite}} \rho_n \in P \,\middle|\, n \in N_\sigma \right\},$$

which is an additive subgroup of $A(N)$. The sum $\sum_{\sigma \in G} A(N)_\sigma$ is direct. Indeed, if we consider a finite sum $\sum_n \rho_n \in A(N)_\sigma \cap \left(\sum_{\tau \in G, \tau \neq \sigma} A(N)_\tau \right)$, then, for any homogeneous element $a \in N_\lambda$, we may write

$$\sum_n \rho_n(a) = \sum_{x \in N_\tau,\, \tau \neq \sigma} \rho_x(a),$$

and thus

$$\sum_n an = \sum_{x \in N_\tau,\, \tau \neq \sigma} ax.$$

Since the left-hand side is an element of $N_{\lambda\sigma}$ and the right-hand side is an element of $\sum_{\lambda\tau,\, \lambda\tau \neq \lambda\sigma} N_{\lambda\tau}$, we obtain that $\sum_n \rho_n(a) = 0$, for all $a \in N_\lambda$. Hence $\sum_n \rho_n(a) = 0$, for all $a \in N$. Therefore, $\sum_n \rho_n = 0$, so

$$A(N)_\sigma \cap \left(\sum_{\tau \in G, \tau \neq \sigma} A(N)_\tau \right) = 0, \text{ for all } \sigma \in G,$$

hence the sum $\sum_{\sigma \in G} A(N)_\sigma$ is direct.

Theorem 3.1. *If G is a left cancellative monoid and $N = \bigoplus_{\sigma \in G} N_\sigma$ is a G-graded abelian near-ring, then the set*

$$A(N)^{gr} = \bigoplus_{\sigma \in G} A(N)_\sigma$$

is a G^{op}-graded ring, where G^{op} is the opposite monoid of G.

Proof. We have proved above that the sum is direct. Let us consider two finite sums $\sum_n \rho_n \in A(N)_\sigma$ (with all $n \in N_\sigma$) and $\sum_m \rho_m \in A(N)_\tau$ (with all $m \in N_\tau$). Then:

$$\left(\sum_n \rho_n\right) \circ \left(\sum_m \rho_m\right) = \sum_m \sum_n \rho_{mn}.$$

Since for any $n \in N_\sigma$ and $m \in N_\tau$ we have $mn \in N_{\tau\sigma}$, it follows that

$$A(N)_\sigma \circ A(N)_\tau \subseteq A(N)_{\tau\sigma}, \text{ for all } \sigma, \tau \in G.$$

Therefore, $A(N)^{gr}$ is a G^{op}-graded ring. □

Definition 3.2. Let $N = \bigoplus_{\sigma\in G} N_\sigma$ be a G-graded abelian near-ring, where G is a left cancellative monoid. The ring $A(N)^{gr} = \bigoplus_{\sigma\in G} A(N)_\sigma$ from Theorem 3.1 is called *the associated graded ring* of N.

Remark 3.3. The following assertions are clearly true:

(i) $A(N)^{gr}$ is a subring of $A(N)$.

(ii) If N is a G-graded ring, then $A(N)^{gr} = A(N)$.

Let $N = \bigoplus_{\sigma\in G} N_\sigma$ be a G-graded abelian near-ring. For any $\sigma \in G$, let

$$\mathrm{END}(N,+)_\sigma = \{f \in \mathrm{End}(N) \mid f(N_\tau) \subseteq N_{\sigma\tau}, \text{ for any } \tau \in G\}$$

which is an additive subgroup of $\mathrm{End}(N,+)$. The sum $\sum_{\sigma\in G} \mathrm{END}(N,+)_\sigma$ is direct, and we denote it by

$$\mathrm{END}(N,+) = \bigoplus_{\sigma\in G} \mathrm{END}(N,+)_\sigma,$$

which is a G-graded ring. Note that if G is finite, then $\mathrm{END}(N,+) = \mathrm{End}(N,+)$. We have the following inclusions (as subrings):

$$\begin{array}{ccc} \mathrm{END}(N,+) & \subseteq & \mathrm{End}(N,+) \\ \cup & & \cup \\ A(N)^{gr} & \subseteq & A(N) \end{array}$$

Moreover, if G is finite, then $A(N)^{gr} \subseteq A(N) \cap \mathrm{END}(N,+)$.

Example 3.4. Consider the near-ring $N = (R_0[X], +, \circ)$ from Example 2.1. The associated graded ring of N is $A(R_0[X])^{gr} = \bigoplus_{n\geq 1} A(R_0[X])_n$, with $A(R_0[X])_n = \{\rho_n \mid n \in RX^n\}$, where

$$\rho_n(a_1X+\cdots+a_kX^k) = (a_1X+\cdots+a_kX^k)\circ X^n = a_1X^n+a_2X^{2n}+\cdots+a_kX^{kn},$$

for any $k \geq 1$ and $a_1X + \cdots + a_kX^k \in R_0[X]$. It follows easily that $A(R_0[X])_n$ is isomorphic to $R_0[X^n]$, for any $n \geq 1$, where we consider X^n as an indeterminate. Therefore, we obtain the isomorphisms

$$A(R_0[X])^{gr} \simeq \bigoplus_{n \geq 1} R_0[X^n] \simeq R_0[Y]^{(\mathbb{N}^*)},$$

where Y is an indeterminate and $R_0[Y]^{(\mathbb{N}^*)}$ denotes a direct sum of copies of $R[Y]$ indexed by the set of nonzero positive integers.

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References

[1] J.R. Clay, *Nearrings: Geneses and Applications*, Oxford University Press, Oxford, 1992.

[2] J. Gutierrez, C. Ruiz De Velasco Y Bellas, *Distributive elements in the near-rings of polynomials*, Proc. Edinburgh Math. Soc. **32** (1989), 73–80.

[3] J.D.P. Meldrum, *Near-Rings and Their Links with Groups*, Pitman Publishing Co., London, 1985.

[4] C. Năstăsescu, F. Van Oystaeyen, *Methods of Graded Rings*, Springer-Verlag, Berlin-Heidelberg, 2004.

[5] G. Pilz, *Near-Rings: The Theory and Its Applications*, Revised Edition, North Holland Publishing Co., Amsterdam, 1983.

[6] K.C. Smith, *A ring associated with a near-ring*, J. Algebra **182** (1996), 329–339.

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