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Ingarden mechanical systems with special external forces

Otilia Lungu and Valer Nimineţ

Abstract

In the present paper we study a remarcable particular case of Finslerian mechanical system, called Ingarden mechanical system. This is defined by a 4-uple $\sum_{IF^n} = \left(M, F^2, N, F_e\right)$ where M is the configuration space, $F^n = (M, F(x,y)) = (M, \alpha(x,y) + \beta(x,y))$ is an Ingarden space, N is the Lorentz nonlinear connection and $F_e = a^i_{jk}(x)\, y^j y^k \frac{\partial}{\partial y^i}$ are the external forces.

One associates to this system $\sum_{IF^n}$ a semispray S, or a dinamical system on the velocity space TM. We write the generalized Maxwell equations for the electromagnetic fields of $\sum_{IF^n}$.

1 Introduction

The general theory of Finslerian mechanical systems was realized by R. Miron [9], [10] and proceeds from the Finsler geometry. It started with Finsler's dissertation in 1918 and its study has been developed by geometers and physicists as: E.Cartan, H.Rund, L.Berwald, S.S.Chern, M.Matsumoto, R.Miron, H.Shimada, G.S.Asanov, etc.

In this paper we introduce and investigate some geometric aspects of a special kind of Finslerian mechanical systems.

We define a 4-uple $\sum_{IF^n} = \left(M,\, F^2,\, N,\, F_e\right)$ where M is the configuration space, $F = \alpha + \beta$ is a Randers metric F^2 is the kinetic energy of the space, N

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is the Lorentz nonlinear connection and $F_e = a^i_{jk}(x)\, y^j y^k \frac{\partial}{\partial y^i}$ are the external forces with $a^i_{jk}(x)$ a symmetric tensor on M of type $(1, 2)$.

We call this 4-uple an Ingarden mechanical system with special external forces and we determine the coefficients of the canonical nonlinear connection $\overset{MI}{N}$. We also construct the canonical metrical d-connection $MI\Gamma(\overset{MI}{N})$ of $\sum_{IF^n}$ and we write the generalized Maxwell equations.

Let M be an n-dimensional, real C^∞ manifold. Denote by (TM, τ, M) the tangent bundle of M and $F^n = (M, F(x, y))$ be a Finsler space, where $F : TM \to R_+$ is its fundamental function, i.e., F verifies the following axioms:

i) F is a differentiable function on $\tilde{TM} = TM\backslash\{0\}$ and it is continuous on the null section of the projection $\tau : TM \to M$;

ii) F is positively 1- homogeneous with respect to the variables y^i;

iii) $\forall (x, y) \in \tilde{TM}$ the Hessian of F^2 with respect y^i is positive defined. Consequently, the d-tensor field $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive defined. It is called the fundamental tensor, or metric tensor of F^n.

This definition can be extended to the case when the fundamental tensor is of constant sygnature, when we imposed the condition $\det(g_{ij}(x, y) \neq 0)$.

It is well known that a Randers metric is a deformation of a Riemannian or pseudo-Riemannian metric $\alpha(x, y) = \sqrt{a_{ij}(x)\, y^i y^j}$, showing the gravitational field, using a 1-form $\beta(x, y) = b_i(x)\, y^i$, representing the electromagnetic field. Randers spaces are Finsler spaces $F^n = (M, F(x, y)) = (M, \alpha(x, y) + \beta(x, y))$ equipped with Cartan nonlinear connection. For F^n, instead of the Cartan nonlinear connection, R. Miron introduced in [8] the Lorentz nonlinear connection N determined by the Lorentz equations of the space F^n with the metric $F(x, y) = \alpha(x, y) + \beta(x, y)$. The local coefficients of N are $N^i_j = \gamma^i_{jk} y^k - F^i_j$, where γ^i_{jk} are the Christoffel symbols of the Riemannian structure $a = a_{ij}(x)\, dx^i \otimes dx^j$ and $F^i_j(x) = a^{is} F_{sj}$, $F_{sj} = \frac{\partial b_s}{\partial x^j} - \frac{\partial b_j}{\partial x^s}$.
The Finsler space $F^n = (M, F(x, y)) = (M, \alpha(x, y) + \beta(x, y))$ equipped with the Lorentz nonlinear connection N is called an Ingarden space. It is denoted $IF^n = (F^n, N)$.

In Preliminaries we give some known results regarding the Lorentz nonlinear connection and Ingarden spaces.

In section 3 we present main results and in section 4 some applications in physical fields.

2 Preliminaries

Let $F^n = (M, F(x, y))$ be a Finsler space with the fundamental function $F(x, y) = \alpha(x, y) + \beta(x, y)$ where $\alpha(x, y) = \sqrt{a_{ij}(x)\, y^i y^j}$ and $\beta(x, y) = b_i(x)\, y^i$; $a = a_{ij}(x)\, dx^i dx^j$ is a pseudo-Riemannian metric on M and it gives

the gravitational part of the metric F ; $b_i(x)$ is an electromagnetic covector on M and $\beta(x, dx) = b_i(x)\, dx^i$ is the electromagnetic 1-form field on M.
We consider the integral of action of the energy $F^2(x, y)$ along a curve $c : t \in [0, 1] \to c(t) \in M$:

$$I(c) = \int_0^1 F^2\left(x, \tfrac{dx}{dt}\right) dt = \int_0^1 \left[\alpha\left(x, \tfrac{dx}{dt}\right) + \beta\left(x, \tfrac{dx}{dt}\right)\right]^2 dt \tag{1}$$

The variational problem for $I(c)$ leads to the Euler-Lagrange equations:

$$E_i\left(F^2\right) := \frac{\partial(\alpha+\beta)^2}{\partial x^i} - \frac{d}{dt}\frac{\partial(\alpha+\beta)^2}{\partial y^i} = 0,\ y^i = \frac{dx^i}{dt}. \tag{2}$$

The energy of F^2 is

$$\varepsilon_{F^2} = y^i \frac{\partial F^2}{\partial y^i} - F^2 = 2F^2 - F^2 = F^2. \tag{3}$$

The covector field $E_i\left(F^2\right)$ is expressed by

$$E_i\left(F^2\right) = E_i\left(\alpha^2\right) + 2\alpha E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial\alpha}{\partial y^i}. \tag{4}$$

Let us fix a parametrization of the curve c, by natural parameter s with respect to Riemannian metric $\alpha(x, y)$. It is given by

$$ds^2 = \alpha^2\left(x, \tfrac{dx}{dt}\right) dt^2. \tag{5}$$

It follows $F^2\left(x, \frac{dx}{ds}\right) = 1$ and $\frac{d\alpha}{ds} = 0$.
Along to an extremal curve c, canonical parametrized by (5), $E_i(\beta)$ is expressed by

$$E_i(\beta) = \left(\frac{\partial b_j}{\partial x^i} - \frac{\partial b_i}{\partial x^j}\right)\frac{dx^j}{ds} = F_{ij}(x)\,\frac{dx^j}{ds}. \tag{6}$$

One obtains [6]:

Theorem 2.1. **(Miron-Hassan)** *In the canonical parametrization, the Euler-Lagrange equations of the Lagrangian* $(\alpha+\beta)^2$ *are given by*

$$E_i\left(\alpha^2\right) + 2F_{ij}(x)\, y^j = 0,\ y^i = \frac{dx^i}{ds}. \tag{7}$$

Theorem 2.2. *The Euler-Lagrange equations* (7) *are equivalent to the Lorentz equations:*

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x)\,\frac{dx^j}{ds}\frac{dx^k}{ds} = \overset{\circ}{F}{}^i_j(x)\,\frac{dx^j}{ds}, \tag{8}$$

where $\overset{\circ}{F}{}^i_j(x) = a^{is}F_{sj}(x)$ *and* γ^i_{jk} *are the Christoffel symbols of the Riemannian metric tensor* $a_{ij}(x)$.

The Euler-Lagrange equations $E_i\left(F^2\right) = 0$ determines a canonical semispray or a Dynamical System S on the total space of the tangent bundle :

$$S = y^i \tfrac{\partial}{\partial x^i} - 2G^i \tfrac{\partial}{\partial y^i}, \tag{9}$$

where the coefficients $G^i\left(x,\, y\right)$ are:

$$2G^i\left(x,\, y\right) = \gamma^i_{jk}\left(x\right)\, y^j y^k - \overset{\circ}{F}{}^i_j\left(x\right)\, y^j. \tag{10}$$

Now we can consider the nonlinear connection N with the coefficients $N^i_j = \frac{\partial G^i}{\partial y^j}$. Of course, we have

$$N^i_j = \gamma^i_{jk}\left(x\right)\, y^k - F^i_j\left(x\right), \tag{11}$$

where $F^i_j\left(x\right) = \frac{1}{2}\overset{\circ}{F}{}^i_j\left(x\right).$

Since the autoparallel curves of N are given by the Lorentz equations (8), we call it the Lorentz nonlinear connection of the Randers metric $\alpha + \beta$.

The nonlinear connection N determines the horizontal distribution, denoted by N too, with the property $T_u TM = N_u \oplus V_u$, $\forall u \in TM$, V_u being the natural vertical distribution on the tangent manifold TM.

The local adapted basis to the horizontal and vertical vector spaces N_u and V_u is given by $\left(\frac{\delta}{\delta x^i},\, \frac{\partial}{\partial y^i}\right)$, $i = 1, ..., n$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i \frac{\partial}{\partial y^k} = \frac{\partial}{\partial x^i} - \gamma^k_{is}\left(x\right) y^s \frac{\partial}{\partial y^k} + F^k_i \frac{\partial}{\partial y^k} = \frac{\overset{\circ}{\delta}}{\delta x^i} + F^k_i \frac{\partial}{\partial y^k} \tag{12}$$

and $\frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \gamma^k_{is}\left(x\right) y^s \frac{\partial}{\partial y^k}$.

The adapted cobasis is $\left(dx^i,\, \delta y^i\right)$, $i = 1, ..., n$ with

$$\delta y^i = dy^i + N^i_j dx^j = dy^i + \gamma^i_{jk}\left(x\right) y^k dx^j - F^i_j dx^j = \overset{\circ}{\delta} y^i - F^i_j dx^j, \tag{13}$$

where $\overset{\circ}{\delta} y^i = dy^i + \gamma^i_{jk}\left(x\right) y^k dx^j$.

The weakly torsion of N is

$$T^i_{jk} = \tfrac{\partial N^i_j}{\partial y^k} - \tfrac{\partial N^i_k}{\partial y^j} = 0. \tag{14}$$

The integrability tensor of N is

$$R^i_{jk} = \tfrac{\delta N^i_j}{\delta x^k} - \tfrac{\delta N^i_k}{\delta x^j}. \tag{15}$$

Definition 2.1. The Finsler space $F^n = (M, F = \alpha + \beta)$ equipped with the Lorentz nonlinear connection N is called an Ingarden space. It is denoted IF^n.

The fundamental tensor g_{ij} of IF^n is

$$g_{ij} = \tfrac{F}{\alpha}(a_{ij} - \tilde{l}_i \tilde{l}_j) + l_i l_j \tag{16}$$

where $\tilde{l}_i = \frac{\partial \alpha}{\partial y^i}$, $l_i = \frac{\partial F}{\partial y^i}$, $l_i = \tilde{l}_i + b_i$.

The following results holds [8]:

Theorem 2.3. *There exists an unique N-metrical connection $I\Gamma(N) = \left(F^i_{jk}, C^i_{jk}\right)$ of the Ingarden space IF^n which verifies the following axioms:*

i) $\nabla^H_k g_{ij} = 0$; $\nabla^V_k g_{ij} = 0$;

ii) $T^i_{jk} = 0$; $S^i_{jk} = 0$.

The connection $I\Gamma(N)$ has the coefficients expressed by the generalized Christoffel symbols:

$$\begin{cases} F^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s} \right) \\ C^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right), \end{cases} \tag{17}$$

where $\frac{\delta}{\delta x^i}$ are given by (12).

3 Main results

For a manifold M, that is the configuration space of a Finslerian dynamical system, let us consider the tangent bundle TM to which we refer to as the velocity space. Suppose that there exists a Randers metric $F = \alpha + \beta$ on $T\tilde{M}$ and $a^i_{jk}(x)$ a symmetric tensor on the configuration space M, of type $(1, 2)$.

Definition 3.1. An Ingarden mechanical system with special external forces is a 4-uple

$$\sum\nolimits_{IF^n} = \left(M,\, (\alpha+\beta)^2,\, N, F_e\right),$$

with N, the Lorentz nonlinear connection and $F_e = a^i_{jk}(x)\, y^j y^k \frac{\partial}{\partial y^i}$ the external forces given as a vertical vector field globaly defined on TM .

We denote $F^i(x,y) = a^i_{jk}(x)\, y^j y^k$and we can state

Theorem 3.1. **[9]** *For the Ingarden mechanical system $\sum_{IF^n} = \left(M,\, (\alpha+\beta)^2,\, N, F_e\right)$ the following properties hold good:*

i) The operator S defined by

$$S = y^i \frac{\partial}{\partial x^i} - \left(2G^i - \tfrac{1}{2} F^i\right) \frac{\partial}{\partial y^i} \tag{18}$$

is a vector field, global defined on the velocity space TM.

ii) S is a semispray which depends only on $\sum_{IF^n}$ and it is a spray if F_e are 2-homogeneous with respect to y^i.

iii) The integral curves of the vector field S are the evolution curves given by the Lagrange equations of $\sum_{IF^n}$:

$$\tfrac{d^2x^i}{dt^2} + \Gamma^i_{jk}\left(x,\ \tfrac{dx}{dt}\right)\ \tfrac{dx^j}{dt}\tfrac{dx^k}{dt} = \tfrac{1}{2}F^i\left(x,\ \tfrac{dx}{dt}\right). \tag{19}$$

The semispray S (18) has the coefficients $\overset{MI}{G^i}$ expressed by

$$2\overset{MI}{G^i} = 2G^i - \frac{1}{2}F^i(x,y) = \Gamma^i_{jk}(x,y)\,y^j y^k - \frac{1}{2}F^i(x,y). \tag{20}$$

Thus, the canonical nonlinear connection $\overset{MI}{N}$ of the Ingarden mechanical system $\sum_{IF^n}$ has the coefficients

$$\overset{MI}{N^i_j} = \frac{\partial\overset{MI}{G^i}}{\partial y^j} = N^i_j - \frac{1}{4}\frac{\partial F^i}{\partial y^j} = N^i_j - \frac{1}{2}a^i_{jk}(x)\,y^k. \tag{21}$$

This nonlinear connection $\overset{MI}{N}$ determines a direct decomposition of the tangent space $T\tilde{M}$ into horizontal and vertical subspaces:

$$T_uT\tilde{M} = \overset{MI}{N_u}\oplus V_u, \forall u \in T\tilde{M}. \tag{22}$$

A local adapted basis to this decomposition is $\left(\frac{\overset{MI}{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i}\right)_{i=\overline{1,n}}$ where

$$\frac{\overset{MI}{\delta}}{\delta x^i} = \frac{\overset{\circ}{\delta}}{\delta x^i} + \left(F^j_i(x) + \frac{1}{2}a^j_{ik}(x)\,y^k\right)\frac{\partial}{\partial y^j} = \frac{\overset{\circ}{\delta}}{\delta x^i} + A^j_i\frac{\partial}{\partial y^j} \tag{23}$$

with

$$A^j_i = F^j_i(x) + \frac{1}{2}a^j_{ik}(x)\,y^k \tag{24}$$

and $\frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \gamma^k_{is}(x)\,y^s\frac{\partial}{\partial y^k}$.

The adapted cobasis is $\left(dx^i, \overset{MI}{\delta}\ y^i\right)$ with

$$\overset{MI}{\delta}\ y^i = \overset{\circ}{\delta}\,y^i - \left(F^i_j + \frac{1}{2}a^i_{jk}(x)\,y^k\right)dx^j = \overset{\circ}{\delta}\,y^i - A^i_j dx^j \tag{25}$$

where $\overset{\circ}{\delta}\,y^i = dy^i + \gamma^i_{jk}(x)\,y^k dx^j$.

We determine the torsion $\overset{MI}{T}{}^i_{jk}$ and the curvature $\overset{MI}{R}{}^i_{jk}$ of the canonical connection $\overset{MI}{N}$ by a direct calculation:

$$\overset{MI}{T}{}^i_{jk} = \frac{\partial \overset{MI}{N}{}^i_j}{\partial y^k} - \frac{\partial \overset{MI}{N}{}^i_k}{\partial y^j} = 0 \tag{26}$$

$$\overset{MI}{R}{}^i_{jk} = \frac{\overset{MI}{\delta} \overset{MI}{N}{}^i_j}{\delta y^k} - \frac{\overset{MI}{\delta} \overset{MI}{N}{}^i_k}{\delta y^j} = \overset{\circ}{R}{}^i_{jk} + \left(A^j_k \frac{\partial N^i_j}{\partial y^j} - A^k_j \frac{\partial N^i_k}{\partial y^k} \right), \tag{27}$$

where we have denoted $\overset{\circ}{R}{}^i_{jk} = \frac{\overset{\circ}{\delta} N^i_j}{\delta x^k} - \frac{\overset{\circ}{\delta} N^i_k}{\delta x^j}$.

Applying the theory from the book [10] the following theorem holds:

Theorem 3.2. *Let $\sum_{IF^n} = \left(M, (\alpha+\beta)^2, F_e\right)$ be an Ingarden mechanical system and $\overset{MI}{N}$ the canonical nonlinear connection of $\sum_{IF^n}$. There exists an unique d-connection $MI\Gamma\left(\overset{MI}{N}\right) = \left(\overset{MI}{F}{}^i_{jk}, \overset{MI}{C}{}^i_{jk}\right)$ determined by the following axioms:*

i) $\overset{MI}{\nabla}{}^H_k g_{ij} = 0$; $\overset{MI}{\nabla}{}^V_k g_{ij} = 0$,

ii) $\overset{MI}{T}{}^i_{jk} = 0$; $\overset{MI}{S}{}^i_{jk} = 0$,

where

$$\begin{array}{l} \overset{MI}{\nabla}{}^H_k g_{ij} = \frac{\overset{MI}{\delta} g_{ij}}{\delta x^k} - \overset{MI}{F}{}^s_{ik} g_{sj} - \overset{MI}{F}{}^s_{jk} g_{is} \\ \overset{MI}{\nabla}{}^V_k g_{ij} = \frac{\partial g_{ij}}{\partial y^k} - \overset{MI}{C}{}^s_{ik} g_{sj} - \overset{MI}{C}{}^s_{jk} g_{is} \end{array}. \tag{28}$$

We call this connection the canonical metrical d-connection of $\sum_{IF^n}$.

Theorem 3.3. *The local coefficients of the canonical metrical d-connection of $\sum_{IF^n}$ are*

$$\begin{cases} \overset{MI}{F}{}^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\overset{MI}{\delta} g_{sj}}{\delta x^k} + \frac{\overset{MI}{\delta} g_{sk}}{\delta x^j} - \frac{\overset{MI}{\delta} g_{jk}}{\delta x^s} \right) \\ \overset{MI}{C}{}^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s} \right). \end{cases} \tag{29}$$

In order to calculate $\overset{MI}{F}{}^i_{jk}$ and $\overset{MI}{C}{}^i_{jk}$ we have:

$$\frac{\overset{MI}{\delta} g_{sj}}{\delta x^k} = \frac{\overset{\circ}{\delta} g_{sj}}{\delta x^k} + A^r_k \frac{\partial g_{sj}}{\partial y^r} \tag{30}$$

Denote $\overset{\circ}{\underset{k}{\nabla}}$ the h-covariant derivative with respect to Levi-Civita connection:

$$\overset{\circ}{\underset{k}{\nabla}} g_{sj} = \frac{\overset{\circ}{\delta} g_{sj}}{\delta x^k} - \gamma^i_{sk} g_{ij} - \gamma^i_{jk} g_{si}. \tag{31}$$

We get

$$\frac{\overset{\circ}{\delta} g_{sj}}{\delta x^k} = \overset{\circ}{\underset{k}{\nabla}} g_{sj} + \gamma^i_{sk} g_{ij} + \gamma^i_{jk} g_{si} \tag{32}$$

Now we obtain

$$\frac{\overset{MI}{\delta} g_{sj}}{\delta x^k} = \overset{\circ}{\underset{k}{\nabla}} g_{sj} + \gamma^i_{sk} g_{ij} + \gamma^i_{jk} g_{si} + A^r_k \frac{\partial g_{sj}}{\partial y^r} \tag{33}$$

and we can state:

Theorem 3.4. *The canonical metrical d-connection of $\sum_{IF^n}$ has the coefficients*

$$\begin{cases} \overset{MI}{F^i_{jk}} = \gamma^i_{jk} + B^i_{jk} \\ \overset{MI}{C^i_{jk}} = C^i_{jk}, \end{cases} \tag{34}$$

where

$$B^i_{jk} = \frac{1}{2} g^{is} \left[\left(\overset{\circ}{\underset{k}{\nabla}} g_{sj} + A^r_k \frac{\partial g_{sj}}{\partial y^r} \right) + \left(\overset{\circ}{\underset{j}{\nabla}} g_{sk} + A^r_j \frac{\partial g_{sk}}{\partial y^r} \right) - \left(\overset{\circ}{\underset{s}{\nabla}} g_{jk} + A^r_s \frac{\partial g_{jk}}{\partial y^r} \right) \right]. \tag{35}$$

Taking into account (33) we can express the curvature tensors of $MI\Gamma\left(\overset{MI}{N}\right) = \left(\overset{MI}{F^i_{jk}}, \overset{MI}{C^i_{jk}}\right)$:

$$\begin{cases} \overset{MI}{R^i_{jkh}} = \frac{\overset{MI}{\delta} \overset{MI}{F^i_{jk}}}{\delta x^h} - \frac{\overset{MI}{\delta} \overset{MI}{F^i_{jh}}}{\delta x^k} + \overset{MI}{F^s_{jk}} \overset{MI}{F^i_{sh}} - \overset{MI}{F^s_{jh}} \overset{MI}{F^i_{sk}} + \overset{MI}{C^i_{hs}} \overset{MI}{R^s_{kh}} \\ \overset{MI}{P^i_{jkh}} = \frac{\partial \overset{MI}{F^i_{jk}}}{\partial y^h} - \overset{MI}{\nabla^H_k} \overset{MI}{C^i_{hs}} + \overset{MI}{C^i_{js}} \overset{MI}{P^s_{kh}} \\ \overset{MI}{S^i_{jkh}} = \frac{\partial \overset{MI}{C^i_{jk}}}{\partial y^h} - \frac{\partial \overset{MI}{C^i_{jh}}}{\partial y^k} + \overset{MI}{C^s_{jk}} \overset{MI}{C^i_{sh}} - \overset{MI}{C^s_{jh}} \overset{MI}{C^i_{sk}} \end{cases} \tag{36}$$

with $\overset{MI}{P^i_{jk}} = \frac{\partial \overset{MI}{N^i_j}}{\partial y^k} - \overset{MI}{F^i_{jk}}$.

4 Applications in physical fields

In an Ingarden mechanical system the h-deflection tensor $\overset{MI}{D}{}^i_k$of the canonical metrical connection no vanishes. It give rise to an interior electromagnetic tensor which is not coincident to the exterior electromagnetic tensor $F_{ik}(x)$ provided by β.

The h-deflection tensor $\overset{MI}{D}{}^i_k$ is given by

$$\overset{MI}{D}{}^i_k = \overset{MI}{\nabla}{}^H_k y^i = \frac{\overset{MI}{\delta}\ y^i}{\delta x^k} + \overset{MI}{F}{}^i_{kj} y^j = B^i_{jk} y^j + A^i_k \tag{37}$$

From the relation

$$B^i_{jk} y^j = \frac{1}{2} g^{is} y^j \left(\underset{k}{\overset{\circ}{\nabla}} g_{sj} + \underset{j}{\overset{\circ}{\nabla}} g_{sk} - \underset{s}{\overset{\circ}{\nabla}} g_{jk} \right) + \frac{1}{2} g^{is} A^r_j \frac{\partial g_{sk}}{\partial y^r} y^j \tag{38}$$

we get

$$\overset{MI}{D}{}^i_k = \frac{1}{2} g^{is} y^j \left(\underset{k}{\overset{\circ}{\nabla}} g_{sj} + \underset{j}{\overset{\circ}{\nabla}} g_{sk} - \underset{s}{\overset{\circ}{\nabla}} g_{jk} \right) + \frac{1}{2} g^{is} A^r_j \frac{\partial g_{sk}}{\partial y^r} y^j + A^i_k. \tag{39}$$

The v-deflection tensor $\overset{MI}{d}{}^i_k$ is

$$\overset{MI}{d}{}^i_k = \overset{MI}{\nabla}{}^V_k y^i = \delta^i_k. \tag{40}$$

The covariant h-tensor is

$$\overset{MI}{D}_{sk} = g_{is} \overset{MI}{D}{}^i_k = \frac{1}{2} y^j \left(\underset{k}{\overset{\circ}{\nabla}} g_{sj} + \underset{j}{\overset{\circ}{\nabla}} g_{sk} - \underset{s}{\overset{\circ}{\nabla}} g_{jk} \right) + \frac{1}{2} A^r_j \frac{\partial g_{sk}}{\partial y^r} y^j + g_{is} A^i_k. \tag{41}$$

and the covariant v-tensor is

$$\overset{MI}{d}_{sk} = g_{is} \overset{MI}{d}{}^s_k\ . \tag{42}$$

The h-interior electromagnetic tensor $\underset{sk}{\overset{\approx}{F}}$ is

$$\overset{\approx}{F}_{sk} = \frac{1}{2} \left(\overset{MI}{D}_{sk} - \overset{MI}{D}_{ks} \right). \tag{43}$$

and the v-interior electromagnetic tensor $\overset{\approx}{f}_{sk}$ is

$$\overset{\approx}{f}_{sk} = \frac{1}{2}\left(\overset{MI}{d}_{sk} - \overset{MI}{d}_{ks}\,.\right) \tag{44}$$

A direct calculus allows to formulate:

Theorem 4.1. *The h- and v- interior electromagnetic tensors of the Ingarden mechanical system $\sum_{IF^n}$ with respect to the canonical metrical connection $\overset{MI}{N}$ are given by*

$$\begin{aligned} &\overset{\approx}{F}_{sk} = \tfrac{1}{2} y^j \left(\overset{\circ}{\nabla}_k\, g_{sj} - \overset{\circ}{\nabla}_s\, g_{jk}\right) + \tfrac{1}{2}\left(g_{is} A^i_k - g_{ik} A^i_s\right) \\ &\overset{\approx}{f}_{sk} = 0. \end{aligned} \tag{45}$$

We denote $\overset{MI}{R}_{ijk} = g_{is}\overset{MI}{R^s_{jk}}$, $\overset{MI}{R}_{ijkh} = g_{js}\overset{MI}{R^s_{ikh}}$, $\overset{MI}{P}_{ijk} = g_{is}\overset{MI}{P^s_{jk}}$, $\overset{MI}{P}_{ijkh} = g_{js}\overset{MI}{P^s_{ikh}}$.

By a direct calculus one proves:

Theorem 4.2. *The h- interior electromagnetic tensors $\overset{\approx}{F}_{ij}$ of the Ingarden mechanical system $\sum_{IF^n}$satisfies the following generalized Maxwell equations:*

$$\begin{aligned} &\overset{MI}{\nabla^H_k}\,\overset{\approx}{F}_{ij} + \overset{MI}{\nabla^H_i}\,\overset{\approx}{F}_{jk} + \overset{MI}{\nabla^H_j}\,\overset{\approx}{F}_{ki} = \tfrac{1}{2}\left\{y^r\left(\overset{MI}{R}_{rijk} + \overset{MI}{R}_{rjki} + \overset{MI}{R}_{rkij}\right) - \left(\overset{MI}{R}_{ijk} + \overset{MI}{R}_{jki} + \overset{MI}{R}_{kij}\right)\right\} \\ &\overset{MI}{\nabla^V_k}\,\overset{\approx}{F}_{ij} + \overset{MI}{\nabla^V_i}\,\overset{\approx}{F}_{jk} + \overset{MI}{\nabla^V_j}\,\overset{\approx}{F}_{ki} = \tfrac{1}{2}\left\{y^r\left[\left(\overset{MI}{P}_{rijk} - \overset{MI}{P}_{rikj}\right) + \left(\overset{MI}{P}_{rjki} - \overset{MI}{P}_{rjik}\right) + \left(\overset{MI}{P}_{rkij} - \overset{MI}{P}_{rkji}\right)\right]\right\} \end{aligned} \tag{46}$$

Conclusions. We defined in this paper a new kind of mechanical systems, called Ingarden mechanical system with special external forces. We developed the theory using the geometrical objects fields of the canonical metrical d-connection. After the calculation of the h- and v-interior electromagnetic tensors, we got a new form for the generalized Maxwell equations. The same theory can be also used to write the Einstein equation for the gravitational fields.

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Otilia LUNGU,
"Vasile Alecsandri" University
Department of Mathematics and Informatics
600115
Bacău
Romania
Email: otilia.lungu@ub.ro

Valer NIMINEŢ,
"Vasile Alecsandri" University
Department of Mathematics and Informatics
600115
Bacău
Romania
Email: valern@ub.ro