



A new class of generalized contraction using \mathcal{P} -functions in ordered metric spaces

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Abstract

In this paper, we introduced and studied a new class of mappings in ordered metric spaces that is inspired from the concept of a \mathcal{P} -function introduced in Chaipunya et. al. [10]. With our new class, we furnish fixed point theorems for continuous, noncontinuous, monotonic, and nonmonotonic mappings in various kinds of the ordering structures.

1 Introduction and Preliminaries

Fixed point theory, one of the cornerstone tools in nonlinear functional analysis, has an extensive possible applications in many positive research fields. Banach contraction mapping principle, also known as Banach fixed point theorem, is one of the initial and fundamental results in the metric fixed point theory. This celebrated result of Banach have been generalized and extended by changing the properties of the mappings in various abstract spaces. Here, we mention only a few of them which are related with our work, [2, 3, 4, 5, 6, 7, 8, 9, 16, 17, 19, 20, 21, 22].

One of the remarkable generalization of Banach fixed point theorem was given in 1997 by Alber and Guerre-Delabriere [1] by introducing the notion of weak contraction in the context of Hilbert space. Rhoades [18] considered

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such contractions in the setting of complete metric spaces. We state Rhoades's result in the following:

A mapping $f : X \rightarrow X$, where (X, d) is a metric space, is said to be *weakly contractive* if

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)) \quad (1)$$

for all $x, y \in X$ and $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a function satisfying:

- (i) φ is continuous and nondecreasing;
- (ii) $\varphi(t) = 0$ if and only if $t = 0$;
- (iii) $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$.

It is clear that the contraction condition (1) reduces to an ordinary contraction when $\varphi(t) := kt$, where $0 \leq k < 1$.

Theorem 1.1 ([18]). *Let (X, d) be a complete metric space and f be a weakly contractive mapping. Then f has a unique fixed point x^* in X .*

In 2009, Harjani and Sadarangani [12] considered the result of Rhoades [18] in the setting of partially ordered metric spaces. For the familiarity of the readers, we recollect the necessary notions and definitions to set up partially ordered metric space. A relation \sqsubseteq is a partial ordering on a set X if it is reflexive, antisymmetric and transitive. Here, we write $b \supseteq a$ instead of $a \sqsubseteq b$ to emphasize some particular cases. Any pair $a, b \in X$ is said to be *comparable* if $a \supseteq b$ or $a \sqsubseteq b$. If a set X has a partial ordering \sqsubseteq , we say that it is a *partially ordered set (w.r.t. \sqsubseteq)* and denote it by (X, \sqsubseteq) . A partially ordered set (X, \sqsubseteq) is said to be a *totally ordered set* if any two elements in X are comparable. Additionally, (X, \sqsubseteq) is said to be a *sequentially ordered set* if each element of a convergent sequence in X is comparable with its limit. Furthermore, if (X, d) is a metric space and (X, \sqsubseteq) is a partially ordered (totally ordered, sequentially ordered) set, we say that X is a *partially ordered (totally ordered, sequentially ordered, respectively) metric space*, and will be denoted by (X, \sqsubseteq, d) .

Now, we recall the result proved in [12] :

Theorem 1.2 ([12]). *Let (X, \sqsubseteq, d) be a complete partially ordered metric space and let $f : X \rightarrow X$ be a continuous and nondecreasing mapping such that*

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y))$$

for $x \sqsubseteq y$, where $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a function satisfying:

- (i) φ is continuous and nondecreasing;
- (ii) $\varphi(t) = 0$ if and only if $t = 0$;

(iii) $\lim_{t \rightarrow +\infty} \varphi(t) = +\infty$.

If there exists $x_0 \in X$ such that $x_0 \sqsubseteq f x_0$, then f has a fixed point.

Notice that Harjini and Sadarangani [12] also proved fixed point theorems for noncontinuous mappings, nonincreasing mappings and even for non-monotonic mappings.

Recently, Chaipunya et al. [10] has introduced and studied the notion of a \mathcal{P} -function. They actually investigated a new class of generalized contraction using such \mathcal{P} -functions, which turns out to cover the above-mentioned results and to open a new direction of auxiliary functions used in generalizing the concept of a contraction. Let us recall now the notions and results stated in [10].

Definition 1.3 ([10]). Let (X, \sqsubseteq, d) be a partially ordered metric space. A function $\varrho : X \times X \rightarrow \mathbb{R}$ is called a \mathcal{P} -function w.r.t. \sqsubseteq in X if it satisfies the following conditions:

- (i) $\varrho(x, y) \geq 0$ for every comparable $x, y \in X$;
- (ii) for any sequences $\{x_n\}_{n=1}^{+\infty}, \{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n \rightarrow +\infty} x_n = x$ and $\lim_{n \rightarrow +\infty} y_n = y$, then $\lim_{n \rightarrow +\infty} \varrho(x_n, y_n) = \varrho(x, y)$;
- (iii) for any sequences $\{x_n\}_{n=1}^{+\infty}, \{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if $\lim_{n \rightarrow +\infty} \varrho(x_n, y_n) = 0$ then $\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$.

If, in addition, the following condition is also satisfied:

- (A) for any sequences $\{x_n\}_{n=1}^{+\infty}, \{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n \in \mathbb{N}$, if the limit $\lim_{n \rightarrow +\infty} d(x_n, y_n)$ exists, then the limit $\lim_{n \rightarrow +\infty} \varrho(x_n, y_n)$ also exists,

then ϱ is said to be a \mathcal{P} -function of type (A) w.r.t. \sqsubseteq in X .

Example 1.4 ([10]). Let (X, \sqsubseteq, d) be a partially ordered metric space. Suppose that the function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is defined as in Theorem 1.2. Then, $\varphi \circ d$ is a \mathcal{P} -function of type (A) w.r.t. \sqsubseteq in X .

Proposition 1.5 ([10]). Let (X, \sqsubseteq, d) be a partially ordered metric space and $\varrho : X \times X \rightarrow \mathbb{R}$ be a \mathcal{P} -function w.r.t. \sqsubseteq in X . If $x, y \in X$ are comparable and $\varrho(x, y) = 0$, then $x = y$.

Corollary 1.6 ([10]). Let (X, \sqsubseteq, d) be a totally ordered metric space and $\varrho : X \times X \rightarrow \mathbb{R}$ be a \mathcal{P} -function w.r.t. \sqsubseteq in X . If $x, y \in X$ and $\varrho(x, y) = 0$, then $x = y$.

Example 1.7 ([10]). Let $X = \mathbb{R}$. Define $d, \varrho : X \times X \rightarrow \mathbb{R}$ with $d(x, y) = |x - y|$ and $\varrho(x, y) = 1 + |x - y|$. If X is endowed with a usual ordering \leq , then (X, \leq, d) is a totally ordered metric space with ϱ as a \mathcal{P} -function of type (A) w.r.t. \leq in X . Note that $\varrho(x, y) \neq 0$ for all $x, y \in X$, even when $x = y$.

This example shows that the converses of Proposition 1.5 and Corollary 1.6 are not generally true.

Definition 1.8 ([10]). Let (X, \sqsubseteq, d) be a partially ordered metric space, a mapping $f : X \rightarrow X$ is called a \mathcal{P} -contraction with respect to \sqsubseteq if there exists a \mathcal{P} -function $\varrho : X \times X \rightarrow \mathbb{R}$ with respect to \sqsubseteq in X such that

$$d(fx, fy) \leq d(x, y) - \varrho(x, y) \quad (2)$$

for any comparable $x, y \in X$. Naturally, if there exists a \mathcal{P} -function of type (A) with respect to \sqsubseteq in X such that the inequality (2) holds for any comparable $x, y \in X$, then f is said to be a \mathcal{P} -contraction of type (A) with respect to \sqsubseteq .

Remark 1.9. From Example 1.4, it follows that in partially ordered metric spaces, a weak contraction is also a \mathcal{P} -contraction of type (A).

On the other hand, recall that a self-mapping f on a metric space (X, d) is said to be a *Chatterjea contraction* (defined by Chatterjea in [11]) if there is a constant $k \in [0, \frac{1}{2})$ such that

$$d(fx, fy) \leq k[d(x, fy) + d(y, fx)]$$

for all $x, y \in X$.

The goal of this manuscript is to use the concept of a \mathcal{P} -function to settle a generalization of a Chatterjea contraction in ordered metric spaces.

2 Generalized Chatterjea contraction using \mathcal{P} -functions

In this section, we introduce the concept of a \mathcal{P} -C-contraction (\mathcal{P} -Chatterjea-contraction). Throughout the paper, we assume that \mathbb{R} represents the set of all real numbers while \mathbb{N} represents the set of all positive integers.

Definition 2.1. Let (X, \sqsubseteq, d) be a partially ordered metric space, a mapping $f : X \rightarrow X$ is called a \mathcal{P} -C-contraction with respect to \sqsubseteq if there exists a \mathcal{P} -function $\varrho : X \times X \rightarrow \mathbb{R}$ with respect to \sqsubseteq in X such that the following inequality holds for each comparable $x, y \in X$:

$$d(fx, fy) \leq \frac{d(x, fy) + d(y, fx)}{2} - \varrho(x, y), \quad (3)$$

where

$$Q(x, y) = \max\{\varrho(x, fy), \varrho(y, fx)\} \quad (4)$$

Naturally, if there exists a \mathcal{P} -function of type (A) with respect to \sqsubseteq in X such that the inequality (3) holds for any comparable $x, y \in X$, then f is said to be a \mathcal{P} -C-contraction of type (A) with respect to \sqsubseteq .

2.1 Fixed point theorems for monotonic mappings

Theorem 2.2. *Let (X, \sqsubseteq, d) be a complete partially ordered metric space and $f : X \rightarrow X$ be a continuous and nondecreasing \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq . If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X .*

Proof. Choose $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$. If $fx_0 = x_0$, then the proof is finished. Suppose that $fx_0 \neq x_0$. We define a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since $x_0 \sqsubseteq fx_0$ and f is nondecreasing w.r.t. \sqsubseteq , we obtain

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots$$

Assume that $\varrho(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$, otherwise we can find $n_0 \in \mathbb{N}$ with $x_{n_0} = x_{n_0+1}$, that is $x_{n_0} = fx_{n_0}$ and there is nothing to prove. Hence, we consider only in the case of which $0 < \varrho(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Since $x_n \sqsubseteq x_{n+1}$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) & (5) \\ &\leq \frac{d(x_{n-1}, fx_n) + d(fx_{n-1}, x_n)}{2} - Q(x_{n-1}, x_n) \\ &= \frac{d(x_{n-1}, x_{n+1})}{2} - Q(x_{n-1}, x_n) \\ &\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} - Q(x_{n-1}, x_n), \end{aligned}$$

where

$$Q(x_{n-1}, x_n) = \max\{\varrho(x_{n-1}, x_{n+1}), \varrho(x_n, x_n)\}$$

for all $n \in \mathbb{N}$. Since $\varrho(x, y) \geq 0$ for every comparable $x, y \in X$ then $Q(x_{n-1}, x_n) \geq 0$. Therefore (5) becomes

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{d(x_{n-1}, x_{n+1})}{2} \\ &\leq \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2}. \end{aligned} \quad (6)$$

Thus

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. Therefore, we have $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ nonincreasing. Since $\{d(x_n, x_{n+1})\}_{n=1}^{+\infty}$ is bounded, there exists $l \geq 0$ such that

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = l. \quad (7)$$

Letting $n \rightarrow +\infty$ in (6) we have

$$l \leq \lim_{n \rightarrow +\infty} \frac{d(x_{n-1}, x_{n+1})}{2} \leq \frac{l+l}{2}$$

or, equivalently,

$$\lim_{n \rightarrow +\infty} d(x_{n-1}, x_{n+1}) = 2l. \quad (8)$$

Thus, there exists $q \geq 0$ such that $\lim_{n \rightarrow +\infty} \varrho(x_{n-1}, x_{n+1}) = q$ and then

$$\lim_{n \rightarrow +\infty} Q(x_{n-1}, x_n) \geq q. \quad (9)$$

Again, making $n \rightarrow +\infty$ in (5) and using (7), (8) and (9) we obtain

$$l \leq \frac{1}{2}2l - q$$

Assume that $l > 0$. Then $q = 0$, which implies that $2l = 0$, a contradiction. Therefore, we have

$$\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0. \quad (10)$$

Now we show that $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence in X . Assume the contrary. Then, there exists $\epsilon_0 > 0$ for which we can construct two subsequences $\{x_{m_k}\}_{k=1}^{+\infty}$ and $\{x_{n_k}\}_{k=1}^{+\infty}$ of $\{x_n\}_{n=1}^{+\infty}$ such that n_k is minimal in the sense that $n_k > m_k > k$ and $d(x_{m_k}, x_{n_k}) \geq \epsilon_0$. Therefore, $d(x_{m_k}, x_{n_k-1}) < \epsilon_0$. Observe that

$$\begin{aligned} \epsilon_0 &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &\leq 2d(x_{m_k}, x_{m_k-1}) + d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \\ &< 2d(x_{m_k}, x_{m_k-1}) + \epsilon_0 + d(x_{n_k-1}, x_{n_k}). \end{aligned}$$

Letting $k \rightarrow +\infty$ and using (10), we get

$$\lim_{k \rightarrow +\infty} d(x_{m_k}, x_{n_k}) = \lim_{k \rightarrow +\infty} d(x_{m_k-1}, x_{n_k}) = \lim_{k \rightarrow +\infty} d(x_{m_k}, x_{n_k-1}) = \epsilon_0. \quad (11)$$

Furthermore, we deduce that the limit $\lim_{k \rightarrow +\infty} \varrho(x_{m_k-1}, x_{n_k})$ and $\lim_{k \rightarrow +\infty} \varrho(x_{m_k}, x_{n_k-1})$ also exist. Now, by the \mathcal{P} -C-contractivity, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &= d(fx_{m_k-1}, fx_{n_k-1}) \\ &\leq \frac{d(x_{m_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{m_k})}{2} - Q(x_{m_k-1}, x_{n_k-1}) \\ &= \frac{d(x_{m_k-1}, x_{n_k}) + d(x_{n_k-1}, x_{m_k})}{2} - \max\{\varrho(x_{m_k-1}, x_{n_k}), \varrho(x_{n_k-1}, x_{m_k})\}. \end{aligned}$$

From (10) and (11), we may find that

$$0 \leq - \lim_{k \rightarrow +\infty} \max\{\varrho(x_{m_k-1}, x_{n_k}), \varrho(x_{n_k-1}, x_{m_k})\},$$

which further implies that

$$\lim_{k \rightarrow +\infty} \varrho(x_{m_k-1}, x_{n_k}) = 0.$$

This ends up with a contradiction. So, $\{x_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since X is complete, there exists x^* such that $x_n = f^n x_0 \rightarrow x^*$ as $n \rightarrow +\infty$. Finally the continuity of f and $ff^n x_0 = f^{n+1} x_0 \rightarrow x^*$ imply that $fx^* = x^*$. Therefore, x^* is a fixed point of f . \square

Next, we drop the continuity of f in the Theorem 2.2, and find out that we can still guarantee a fixed point if we strengthen the condition of a partially ordered set to a sequentially ordered set.

Theorem 2.3. *Let (X, \sqsubseteq, d) be a complete sequentially ordered metric space and $f : X \rightarrow X$ be a nondecreasing \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq . If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X .*

Proof. If we take $x_n = f^n x_0$ in the proof of Theorem 2.2, then we conclude that $\{x_n\}_{n=1}^{+\infty}$ converges to a point x^* in X .

Next, we prove that x^* is a fixed point of f in X . Indeed, suppose that x^* is not a fixed point of f , i.e., $d(x^*, fx^*) \neq 0$. Since x^* is comparable with x_n for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(x_{n+1}, fx^*) &= d(fx_n, fx^*) \\ &\leq \frac{d(x_n, fx^*) + d(x^*, fx_n)}{2} - Q(x_n, x^*) \\ &= \frac{d(x_n, fx^*) + d(x^*, x_{n+1})}{2} - \max\{\varrho(x_n, fx^*), \varrho(x^*, x_{n+1})\} \\ &\leq \frac{d(x_n, fx^*) + d(x^*, x_{n+1})}{2} \end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \rightarrow +\infty$ we obtain

$$d(x^*, fx^*) \leq \frac{d(x^*, fx^*)}{2}$$

and this is a contradiction unless $d(x^*, fx^*) = 0$, or, equivalently, x^* is a fixed point of f . \square

We give a sufficient condition for the uniqueness of the fixed point in the next Theorem.

Theorem 2.4. *Let (X, \sqsubseteq, d) be a complete partially ordered metric space and $f : X \rightarrow X$ be a continuous and nondecreasing \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq . Suppose that for each $x, y \in X$, there exists $w \in X$ which is comparable to both x and y . If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X .*

Proof. By the Theorem 2.2, we conclude that f has a fixed point. Next, we show that the fixed point of f is unique. Assume that u and v be two distinct fixed points of f , i.e., $d(u, v) \neq 0$. Here, we divide our proof into two cases:

Case 1. If u is comparable to v then $f^n u$ is comparable to $f^n v$ for all $n \in \mathbb{N}$ and

$$\begin{aligned} d(u, v) &= d(f^n u, f^n v) \\ &\leq \frac{d(f^{n-1} u, f^{n-1} v) + d(f^{n-1} v, f^{n-1} u)}{2} - Q(f^{n-1} u, f^{n-1} v) \\ &= \frac{d(u, v) + d(v, u)}{2} - Q(u, v) \\ &= d(u, v) - \max\{\varrho(u, v), \varrho(v, u)\} \\ &\leq d(u, v) - \varrho(u, v). \end{aligned}$$

Since $\varrho(u, v) \geq 0$ then $\varrho(u, v) = 0$, and by definition, $u = v$.

Case 2. If u is not comparable to v then there exist w comparable to u and v . Monotonicity of f implies that $f^n w$ is comparable to $f^n u = u$ and $f^n v = v$ for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} d(u, f^n w) &\leq \frac{d(u, f^n w) + d(f^{n-1} w, u)}{2} - Q(u, f^{n-1} w) \\ &= \frac{d(u, f^n w) + d(f^{n-1} w, u)}{2} - \max\{\varrho(u, f^n w), \varrho(f^{n-1} w, u)\} \\ &\leq \frac{d(u, f^n w) + d(f^{n-1} w, u)}{2} - \varrho(f^{n-1} w, u). \end{aligned} \quad (12)$$

From the above inequality we get

$$d(u, f^n w) \leq d(u, f^{n-1} w)$$

If we define a sequence $s_n = d(u, f^n w)$ and $t_n = \varrho(f^n w, u)$, we may obtain from (12) that $\{s_n\}_{n=1}^{+\infty}$ is nonincreasing and there exists $l, q \geq 0$ such that $\lim_{n \rightarrow +\infty} s_n = l$ and $\lim_{n \rightarrow +\infty} t_n = q$.

Assume that $l > 0$. Then by the \mathcal{P} -C-contractivity of f , we have

$$l \leq l - q.$$

This implies that $q = 0$ and so, this is a contradiction. Hence, $\lim_{n \rightarrow +\infty} s_n = 0$. In the same way, we can also show that $\lim_{n \rightarrow +\infty} d(v, f^n w) = 0$. That is, $\{f^n w\}_{n=1}^{+\infty}$ converges to both u and v . Since the limit of a convergent sequence in a metric space is unique, we conclude that $u = v$. Hence, this yields the uniqueness of the fixed point. \square

Theorem 2.5. *Let (X, \sqsubseteq, d) be a complete sequentially ordered metric space and $f : X \rightarrow X$ be a nondecreasing \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq . Suppose that for each $x, y \in X$, there exists $w \in X$ which is comparable to both x and y . If there exists $x_0 \in X$ with $x_0 \sqsubseteq f x_0$, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X .*

Proof. If we take $x_n = f^n x_0$, then we conclude, by Theorem 2.3, that $\{x_n\}_{n=1}^{+\infty}$ converges to a fixed point of f in X . The rest of the proof is similar to the proof of Theorem 2.4. \square

Remark 2.6. Notice that if (X, \sqsubseteq, d) is a totally ordered set, any two elements in X are comparable and we obtain uniqueness of the fixed point.

2.2 Fixed point theorems for nonmonotonic mappings

In this section, we drop the monotonicity conditions of f and finds out that we still can apply our results to confirm the existence and uniqueness of fixed point of f .

Theorem 2.7. *Let (X, \sqsubseteq, d) be a complete partially ordered metric space and $f : X \rightarrow X$ be a continuous \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq such that the comparability of $x, y \in X$ implies the comparability of $f x, f y \in f X$. If there exists $x_0 \in X$ such that x_0 and $f x_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X .*

Proof. Choose $x_0 \in X$ such that x_0 and $f x_0$ are comparable. If $f x_0 = x_0$, then the proof is finished. Suppose that $f x_0 \neq x_0$. We define a sequence

$\{x_n\}_{n=1}^{+\infty}$ such that $x_n = f^n x_0$. Since x_0 and $f x_0$ are comparable, we have x_n and x_{n+1} comparable for all $n \in \mathbb{N}$. Now, the rest of the proof is similar to the proof of Theorem 2.2. \square

Further results can be proved using the same plots of the earlier theorems in this paper, so we are omitting them.

Theorem 2.8. *Let (X, \sqsubseteq, d) be a complete sequentially ordered metric space and $f : X \rightarrow X$ be a \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq such that the comparability of $x, y \in X$ implies the comparability of $f x, f y \in f X$. If there exists $x_0 \in X$ such that x_0 and $f x_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a fixed point of f in X .*

Theorem 2.9. *Let (X, \sqsubseteq, d) be a complete partially ordered metric space and $f : X \rightarrow X$ be a continuous \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq such that the comparability of $x, y \in X$ implies the comparability of $f x, f y \in f X$. Suppose that each $x, y \in X$, there exists $w \in X$ which is comparable to both x and y . If there exists $x_0 \in X$ such that x_0 and $f x_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X .*

Theorem 2.10. *Let (X, \sqsubseteq, d) be a complete sequentially ordered metric space and $f : X \rightarrow X$ be a \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq such that the comparability of $x, y \in X$ implies the comparability of $f x, f y \in f X$. Suppose that each $x, y \in X$, there exists $w \in X$ which is comparable to both x and y . If there exists $x_0 \in X$ such that x_0 and $f x_0$ are comparable, then $\{f^n x_0\}_{n=1}^{+\infty}$ converges to a unique fixed point of f in X .*

3 Example

Example 3.1. Let $X = \{(0, 1), (1, 0), (1, 1)\} \subset \mathbb{R}^2$ and suppose that we write $x = (x_1, x_2)$ and $y = (y_1, y_2)$ for $x, y \in X$. Define $d, \varrho : X \times X \rightarrow \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 2 \max\{x_1 + y_1, x_2 + y_2\} & \text{otherwise,} \end{cases}$$

and

$$\varrho(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x_1, x_2 + y_2\} & \text{otherwise.} \end{cases}$$

Let \sqsubseteq be an ordering in X given by $R = \{(x, x); x \in X\} \cup \{(0, 1), (1, 1)\}$. Then, (X, \sqsubseteq, d) is a partially ordered metric space with ϱ as a \mathcal{P} -function of type (A) w.r.t. \sqsubseteq in X .

Now, let T be the operator $T : X \rightarrow X$ defined by $T(0, 1) = (0, 1)$, $T(1, 1) = (0, 1)$ and $T(1, 0) = (1, 0)$. Obviously, T is a continuous and nondecreasing mapping w.r.t. \sqsubseteq since $(0, 1) \leq (1, 1)$ and $T(0, 1) = (0, 1) \sqsubseteq T(1, 1) = (0, 1)$.

Let $x, y \in X$ be comparable w.r.t \sqsubseteq . Consider the following four cases.

- Case 1 : $x = y = (0, 1)$. We get that

$$d(T(0, 1), T(0, 1)) = d((0, 1), T(0, 1)) = \varrho((0, 1), T(0, 1)) = 0.$$

- Case 2 : $x = y = (1, 0)$. We observe that

$$d(T(1, 0), T(1, 0)) = d((1, 0), T(1, 0)) = \varrho((1, 0), T(1, 0)) = 0.$$

- Case 3 : $x = y = (1, 1)$. We derive that

$$d(T(1, 1), T(1, 1)) = 0, \quad d((1, 1), T(1, 1)) = 4, \quad \varrho((1, 1), T(1, 1)) = 2.$$

- Case 4 : $x = (0, 1)$ and $y = (1, 1)$. We find that

$$d(T(0, 1), T(1, 1)) = 0, \quad d((0, 1), T(1, 1)) = 0, \quad d((1, 1), T(0, 1)) = 4,$$

and

$$\varrho((0, 1), T(1, 1)) = 0, \quad \varrho((1, 1), T(0, 1)) = 2.$$

Therefore, the inequality (3) is satisfied for every comparable $x, y \in X$. So, T is a continuous and nondecreasing \mathcal{P} -C-contraction of type (A) w.r.t. \sqsubseteq . As $(0, 1) \sqsubseteq T(0, 1)$, Theorem 2.2 shows that T has a fixed point in X (in this case $(0, 1)$ and $(1, 0)$ are two fixed points of T).

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