



Strong convergence of a composite Halpern-type iteration for a family of nonexpansive mappings in CAT(0) spaces

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Abstract

In this paper, we prove the strong convergence of the composite Halpern-type iteration for a family of nonexpansive mappings in CAT(0) spaces and compare our results with several recent results in this subject. Also, the inexact version of the Halpern iteration is studied in CAT(0) spaces.

1 Introduction and Preliminaries

Let C be a nonempty subset of a metric space (X, d) . $T : C \rightarrow C$ is called nonexpansive if for each $x, y \in C$, $d(Tx, Ty) \leq d(x, y)$. A point $x \in C$ is called a fixed point of T if $Tx = x$. We denote $F(T) := \{x \in C : T(x) = x\}$. Halpern [9] proved the strong convergence of the iteration

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad (1.1)$$

where $u \in C$ is an arbitrary (but fixed) element in C , under suitable assumptions on the control sequence α_n . Also, he showed that the assumptions

$$C1 : \lim_{n \rightarrow \infty} \alpha_n = 0,$$

$$C2 : \sum_{n=1}^{\infty} \alpha_n = \infty,$$

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are necessary for the strong convergence of the iteration (1.1) to a fixed point of T . He also proposed the following open problem:

Are conditions $C1$ and $C2$ sufficient for the strong convergence of the sequence generated by (1.1) to a fixed point of T ?

Lions [14] proved the strong convergence of the Halpern iteration under the conditions $C1, C2$ and $(C3) \lim_{n \rightarrow \infty} \frac{\alpha_{n+1} - \alpha_n}{\alpha_{n+1}^2} = 0$.

Wittmann [20] improved the result of Lions under the assumptions $(C1), (C2)$ and $(C4) \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

An analogue of Wittmann's result was first proved by Reich [16] for Banach spaces with a weakly sequentially continuous duality mapping. Shioji and Takahashi [18] extended Wittmann's result to Banach spaces with uniformly Gateaux differentiable norm. Xu [21, 22] proved the strong convergence of the Halpern iteration provided that the conditions $(C1), (C2)$ and $(C5) \lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$ are satisfied.

Suzuki [19] and Chidume-Chidume [5], independently, proved that the conditions $(C1)$ and $(C2)$ are sufficient for the strong convergence of the following iterative sequence:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(\lambda x_n + (1 - \lambda)Tx_n), \tag{1.2}$$

to a fixed point of T , where T is a nonexpansive mapping on a closed and convex subset C of a Banach space with uniformly Gateaux differentiable norm and $x_1, u \in C$. Therefore Halpern open problem has a positive solution when the nonexpansive operator is a convex combination of the identity operator and another nonexpansive operator. Qin, Su and Shang [15] introduced the composite iteration scheme as follows:

$$\begin{cases} z_n = \gamma_n x_n + (1 - \gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1 - \beta_n)Tz_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases} \tag{1.3}$$

where T is a nonexpansive mapping on a closed and convex subset C of a uniformly smooth Banach space, $F(T) \neq \emptyset$, $u \in C$ is an arbitrary (but fixed) element in C , and $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. They proved the sequence $\{x_n\}$ defined by (1.3) converges strongly to a fixed point of T under appropriate assumptions on the sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$.

The main goal of this paper is to prove the strong convergence of the sequence generated by (1.3) for a family of nonexpansive mappings in CAT(0) spaces. In the sequel, we introduce CAT(0) spaces. Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x, y)] \rightarrow X$ such that $c(0) = x, c(d(x, y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y . When it is unique, this geodesic

segment is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices. A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ is the triangle $\overline{\Delta}(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane \mathbb{R}^2 such that

$$d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j) \text{ for } i, j \in \{1, 2, 3\}.$$

A geodesic space X is a CAT(0) space if for each geodesic triangle Δ in X and its comparison triangle $\overline{\Delta}$ in \mathbb{R}^2 , the CAT(0) inequality

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y})$$

is satisfied for all $x, y \in \Delta$ and $\overline{x}, \overline{y} \in \overline{\Delta}$. It is well-known that a CAT(0) space is an uniquely geodesic space.

There are several examples of CAT(0) spaces among metric structures. For example, any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces (see [2]), \mathbb{R} -trees (see [11]), Euclidean buildings (see [3]), the complex Hilbert ball with a hyperbolic metric (see [8]), and many others. For a thorough discussion of these spaces and their fundamental role in geometry, we refer the reader to Bridson and Haefliger [2].

Fixed-point theory in CAT(0) spaces was first studied by Kirk (see [12, 13]). He showed that every nonexpansive (single-valued) mapping defined on a bounded, closed and convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared. It is worth mentioning that fixed-point theorems in CAT(0) spaces (specially in \mathbb{R} -trees) can be applied to graph theory, biology, and computer science.

In this article, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining x to y such that $d(z, x) = td(x, y)$, $d(z, y) = (1-t)d(x, y)$. If X is a CAT(0) space and $x, y \in X$, then $[x, y] := \{(1-t)x \oplus ty : t \in [0, 1]\}$ and a subset C of X is convex if $[x, y] \subseteq C$, for all $x, y \in C$.

Saejung [17] proved the strong convergence of Halpern's iteration in CAT(0) spaces. He considered the iteration:

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T_n x_n, \tag{1.4}$$

where $\{T_n\}$ is a family of nonexpansive mappings on closed and convex subset C of a CAT(0) space with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $x_1, u \in C$. He proved the strong convergence of this iteration to a common fixed point of $\{T_n\}$, under the conditions $C1, C2$ and $C4$ or $C5$ and an appropriate condition on $\{T_n\}$ which is introduced in (3.1).

In this paper, we prove the strong convergence of the sequence generated by

$$\begin{cases} z_n = \gamma_n x_n \oplus (1 - \gamma_n) T_n x_n, \\ y_n = \beta_n x_n \oplus (1 - \beta_n) T_n z_n, \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) y_n, \end{cases} \quad (1.5)$$

where $\{T_n\}$ is a family of nonexpansive mappings on closed and convex subset C of a complete CAT(0) space X with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, $x_1, u \in C$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ that satisfy the following conditions:

- (A₁) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (A₂) $\alpha_n \rightarrow 0, \beta_n \rightarrow 0, \liminf_n \gamma_n > 0$;
- (A₃) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$;
- (A₄) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{|\beta_{n+1} - \beta_n|}{\alpha_{n+1}} = 0$;
- (A₅) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{|\gamma_{n+1} - \gamma_n|}{\alpha_{n+1}} = 0$.

This result extends the result of [15] in CAT(0) spaces. (1.5) is also a modified Ishikawa iteration. If $\beta_n \equiv 0$ and $\gamma_n \equiv 1$ in (1.5), then we get (1.4) which has been considered by Saejung [17].

On the other hand, if $\gamma_n \equiv 1$, then (1.5) can be viewed as a modified Halpern iteration

$$\begin{cases} y_n = \beta_n x_n \oplus (1 - \beta_n) T_n x_n, \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) y_n. \end{cases} \quad (1.6)$$

In (1.6), if $T_n \equiv T$ then (1.6) has been considered by Cuntavepanit and Panyanak [6] in CAT(0) spaces and by Kim and Xu [10] in Banach spaces.

2 Some Lemmas

The following lemmas are needed in the sequel.

Lemma 2.1. [7] *Let (X, d) be a CAT(0) space. Then, for all $x, y, z \in X$ and all $t \in [0, 1]$:*

- (1) $d^2(tx \oplus (1 - t)y, z) \leq td^2(x, z) + (1 - t)d^2(y, z) - t(1 - t)d^2(x, y)$,
- (2) $d(tx \oplus (1 - t)y, z) \leq td(x, z) + (1 - t)d(y, z)$,

In addition, by (1), we have

$$d(tx \oplus (1 - t)y, tx \oplus (1 - t)z) \leq (1 - t)d(y, z).$$

Lemma 2.2. [4] *If (X, d) is a CAT(0) space. Then, for all $x, y \in X$ and all $t, s \in [0, 1]$, we have*

$$d(tx \oplus (1-t)y, sx \oplus (1-s)y) = |t-s|d(x, y).$$

Recall that a Banach limit is a continuous linear functional μ on l^∞ such that $\|\mu\| = \mu(1, 1, \dots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$, for all $\{a_n\} \in l^\infty$.

Lemma 2.3. [18] *Let (a_1, a_2, \dots) be in l^∞ such that $\mu_n(a_n) \leq 0$ for all Banach limits μ and $\limsup_n(a_{n+1} - a_n) \leq 0$, then $\limsup_n a_n \leq 0$.*

Lemma 2.4. [1] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ a sequence of real numbers in $[0, 1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, $\{u_n\}$ a sequence of nonnegative real numbers with $\sum_{n=1}^\infty u_n < \infty$, and $\{t_n\}$ a sequence of real numbers with $\limsup_n t_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n t_n + u_n,$$

for all $n \in \mathbb{N}$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.5. [17] *Let C be a closed and convex subset of a complete CAT(0) space X and let $T : C \rightarrow C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0, 1)$, the mapping $S_t : C \rightarrow C$ defined by*

$$S_t x = tu \oplus (1-t)Tx, \quad \forall x \in C,$$

has a unique fixed point $x_t \in C$, that is,

$$x_t = S_t x_t = tu \oplus (1-t)Tx_t. \tag{2.1}$$

Lemma 2.6. [17] *Let C be a closed and convex subset of a complete CAT(0) space X and let $T : C \rightarrow C$ be a nonexpansive mapping. Then $F(T) \neq \emptyset$ if and only if $\{x_t\}$ given by the formula (2.1) remains bounded as $t \rightarrow 0$. In this case, the following statements hold:*

- (1) $\{x_t\}$ converges to the unique fixed point z of T , which is the nearest point of $F(T)$ to u ;
- (2) $d^2(u, z) \leq \mu_n d^2(u, x_n)$, for all Banach limits μ and all bounded sequences $\{x_n\}$ with $d(x_n, Tx_n) \rightarrow 0$.

3 Main Results

In the setting of the CAT(0) spaces, we prove the strong convergence of the composite Halpern-type iteration generated by (1.5) and compare our results with other results by some examples. Also, the inexact version of the Halpern iteration is studied.

3.1 Composite Halpern Iteration

The following concept was introduced by Aoyama et al. [1]. Let C be a subset of a metric space X and $\{T_n\}_{n=1}^{\infty} : C \rightarrow C$ is a countable family of mappings. The family $\{T_n\}$ satisfies AKTT-condition if

$$\sum_{n=1}^{\infty} \sup\{d(T_{n+1}z, T_n z) : z \in B\} < \infty, \quad (3.1)$$

for each bounded subset B of C .

If C is a closed subset of X and $\{T_n\}$ satisfies AKTT-condition, then we can define $T : C \rightarrow C$ such that

$$Tx = \lim_{n \rightarrow \infty} T_n x, \quad (x \in C). \quad (3.2)$$

In this case, we also say (T_n, T) satisfies AKTT-condition.

The following theorem extends Theorem 3.2 of Saejung [17].

Theorem 3.1. *Let C be a closed and convex subset of complete CAT(0) space X and let $\{T_n\}_{n=1}^{\infty}$ be a family of nonexpansive mappings of C to itself such that*

$$\begin{cases} (A) (T_n, T) \text{ satisfies AKTT-condition,} \\ (B) F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset. \end{cases} \quad (3.3)$$

Given a point $u \in C$ and the initial guess $x_0 \in C$ is chosen arbitrarily. If $\{x_n\}$ is composite process generated by (1.5), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$ satisfy $A_1 - A_5$ then $\{x_n\}$ converges strongly to $z \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$, which is the nearest point of $F(T)$ to u .

Proof. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. We have

$$\begin{aligned} d(z_n, p) &\leq \gamma_n d(x_n, p) + (1 - \gamma_n) d(T_n x_n, p) \\ &\leq d(x_n, p), \end{aligned}$$

which implies

$$\begin{aligned} d(y_n, p) &\leq \beta_n d(x_n, p) + (1 - \beta_n) d(T_n z_n, p) \\ &\leq \beta_n d(x_n, p) + (1 - \beta_n) d(z_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

This follows that

$$\begin{aligned} d(x_{n+1}, p) &\leq \alpha_n d(u, p) + (1 - \alpha_n) d(y_n, p) \\ &\leq \alpha_n d(u, p) + (1 - \alpha_n) d(x_n, p) \\ &\leq \max\{d(u, p), d(x_n, p)\} \\ &\leq \dots \leq \max\{d(u, p), d(x_0, p)\}. \end{aligned}$$

Thus $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{T_n x_n\}$ and $\{T_n z_n\}$ are bounded. In the sequel, we show $d(x_{n+1}, x_n) \rightarrow 0$. We have

$$\begin{aligned} d(z_n, z_{n-1}) &= d(\gamma_n x_n \oplus (1 - \gamma_n) T_n x_n, \gamma_{n-1} x_{n-1} \oplus (1 - \gamma_{n-1}) T_{n-1} x_{n-1}) \\ &\leq d(\gamma_n x_n \oplus (1 - \gamma_n) T_n x_n, \gamma_n x_n \oplus (1 - \gamma_n) T_n x_{n-1}) \\ &\quad + d(\gamma_n x_n \oplus (1 - \gamma_n) T_n x_{n-1}, \gamma_n x_n \oplus (1 - \gamma_n) T_{n-1} x_{n-1}) \\ &\quad + d(\gamma_n x_n \oplus (1 - \gamma_n) T_{n-1} x_{n-1}, \gamma_{n-1} x_{n-1} \oplus (1 - \gamma_{n-1}) T_{n-1} x_{n-1}) \\ &\quad + d(\gamma_{n-1} x_{n-1} \oplus (1 - \gamma_{n-1}) T_{n-1} x_{n-1}, \gamma_{n-1} x_{n-1} \oplus (1 - \gamma_{n-1}) T_{n-1} x_{n-1}) \\ &\leq (1 - \gamma_n) d(T_n x_n, T_n x_{n-1}) + (1 - \gamma_n) d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\ &\quad + |\gamma_n - \gamma_{n-1}| d(x_n, T_{n-1} x_{n-1}) + \gamma_{n-1} d(x_n, x_{n-1}) \\ &\leq (1 - \gamma_n) d(x_n, x_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\ &\quad + |\gamma_n - \gamma_{n-1}| d(x_n, T_{n-1} x_{n-1}) + \gamma_{n-1} d(x_n, x_{n-1}). \end{aligned}$$

With a similar computation, we get

$$\begin{aligned} d(y_n, y_{n-1}) &\leq (1 - \beta_n) d(z_n, z_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \\ &\quad + |\beta_n - \beta_{n-1}| d(x_n, T_{n-1} z_{n-1}) + \beta_{n-1} d(x_n, x_{n-1}). \end{aligned}$$

Therefore

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n u \oplus (1 - \alpha_n) y_n, \alpha_{n-1} u \oplus (1 - \alpha_{n-1}) y_{n-1}) \\ &\leq d(\alpha_n u \oplus (1 - \alpha_n) y_n, \alpha_n u \oplus (1 - \alpha_n) y_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1 - \alpha_n) y_{n-1}, \alpha_{n-1} u \oplus (1 - \alpha_{n-1}) y_{n-1}) \\ &\leq (1 - \alpha_n) d(y_n, y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, y_{n-1}) \\ &\leq (1 - \alpha_n) ((1 - \beta_n) d(z_n, z_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \\ &\quad + |\beta_n - \beta_{n-1}| d(x_n, T_{n-1} z_{n-1}) + \beta_{n-1} d(x_n, x_{n-1})) + |\alpha_n - \alpha_{n-1}| d(u, y_{n-1}) \end{aligned}$$

$$\begin{aligned}
 &\leq (1 - \alpha_n)((1 - \beta_n)((1 - \gamma_n)d(x_n, x_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1})) \\
 &\quad + |\gamma_n - \gamma_{n-1}|d(x_n, T_{n-1} x_{n-1}) + \gamma_{n-1}d(x_n, x_{n-1})) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \\
 &\quad + |\beta_n - \beta_{n-1}|d(x_n, T_{n-1} z_{n-1}) + \beta_{n-1}d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, y_{n-1}) \\
 &= (1 - \alpha_n)(1 - \beta_n)(1 - \gamma_n)d(x_n, x_{n-1}) + (1 - \alpha_n)(1 - \beta_n)d(T_n x_{n-1}, T_{n-1} x_{n-1}) \\
 &+ (1 - \alpha_n)(1 - \beta_n)|\gamma_n - \gamma_{n-1}|d(x_n, T_{n-1} x_{n-1}) + (1 - \alpha_n)(1 - \beta_n)\gamma_{n-1}d(x_n, x_{n-1}) \\
 &\quad + (1 - \alpha_n)d(T_n z_{n-1}, T_{n-1} z_{n-1}) + (1 - \alpha_n)|\beta_n - \beta_{n-1}|d(x_n, T_{n-1} z_{n-1}) \\
 &\quad + (1 - \alpha_n)\beta_{n-1}d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, y_{n-1}) \\
 &\leq ((1 - \alpha_n)(1 - \beta_n)(1 - \gamma_n) + (1 - \alpha_n)(1 - \beta_n)\gamma_{n-1} \\
 &\quad + (1 - \alpha_n)\beta_{n-1})d(x_n, x_{n-1}) + |\gamma_n - \gamma_{n-1}|d(x_n, T_{n-1} x_{n-1}) \\
 &\quad + |\beta_n - \beta_{n-1}|d(x_n, T_{n-1} z_{n-1}) + |\alpha_n - \alpha_{n-1}|d(u, y_{n-1}) \\
 &\quad + d(T_n x_{n-1}, T_{n-1} x_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \\
 &= (1 - \alpha_n)(1 + \gamma_{n-1} - \gamma_n + \beta_{n-1} - \beta_n + (\gamma_n - \gamma_{n-1})\beta_n)d(x_n, x_{n-1}) \\
 &\quad + |\gamma_n - \gamma_{n-1}|d(x_n, T_{n-1} x_{n-1}) + |\beta_n - \beta_{n-1}|d(x_n, T_{n-1} z_{n-1}) \\
 &\quad + |\alpha_n - \alpha_{n-1}|d(u, y_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \\
 &\leq (1 - \alpha_n)(1 + |\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n| + |\gamma_n - \gamma_{n-1}|\beta_n)d(x_n, x_{n-1}) \\
 &\quad + |\gamma_n - \gamma_{n-1}|d(x_n, T_{n-1} x_{n-1}) + |\beta_n - \beta_{n-1}|d(x_n, T_{n-1} z_{n-1}) \\
 &\quad + |\alpha_n - \alpha_{n-1}|d(u, y_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}) \\
 &\leq (1 - \alpha_n)d(x_n, x_{n-1}) + (2|\gamma_{n-1} - \gamma_n| + |\beta_{n-1} - \beta_n|)d(x_n, x_{n-1}) \\
 &\quad + |\gamma_n - \gamma_{n-1}|d(x_n, T_{n-1} x_{n-1}) + |\beta_n - \beta_{n-1}|d(x_n, T_{n-1} z_{n-1}) \\
 &\quad + |\alpha_n - \alpha_{n-1}|d(u, y_{n-1}) + d(T_n x_{n-1}, T_{n-1} x_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}).
 \end{aligned}$$

Hence

$$\begin{aligned}
 d(x_{n+1}, x_n) &\leq (1 - \alpha_n)d(x_n, x_{n-1}) \\
 &\quad + M(3|\gamma_n - \gamma_{n-1}| + 2|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|) \\
 &\quad + \sup\{d(T_n u, T_{n-1} u) : u \in \{x_n\}\} + \sup\{d(T_n u, T_{n-1} u) : u \in \{z_n\}\},
 \end{aligned}$$

where $M \geq \max\{d(x_n, x_{n-1}), d(x_n, T_{n-1} x_{n-1}), d(x_n, T_{n-1} z_{n-1}), d(u, y_{n-1})\}$.
Now, by Lemma 2.4, the conditions A_1, A_3, A_4, A_5 and the condition A of (3.3) we obtain

$$d(x_{n+1}, x_n) \rightarrow 0. \tag{3.4}$$

On the other hand

$$\begin{aligned}
 d(x_n, T_n x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_n) + d(y_n, T_n z_n) + d(T_n z_n, T_n x_n) \\
 &\leq d(x_n, x_{n+1}) + \alpha_n d(u, y_n) + \beta_n d(x_n, T_n z_n) + d(z_n, x_n) \\
 &= d(x_n, x_{n+1}) + \alpha_n d(u, y_n) + \beta_n d(x_n, T_n z_n) + (1 - \gamma_n)d(x_n, T_n x_n),
 \end{aligned}$$

which implies

$$\gamma_n d(x_n, T_n x_n) \leq d(x_n, x_{n+1}) + \alpha_n d(u, y_n) + \beta_n d(x_n, T_n z_n).$$

This together with (3.4) follow $\lim_{n \rightarrow \infty} \gamma_n d(x_n, T_n x_n) = 0$, which by $\liminf_n \gamma_n > 0$ implies

$$d(x_n, T_n x_n) \rightarrow 0. \quad (3.5)$$

Also, we have

$$\begin{aligned} d(x_n, T x_n) &\leq d(x_n, T_n x_n) + d(T_n x_n, T x_n) \\ &\leq d(x_n, T_n x_n) + \sup\{d(T_n y, T y) : y \in \{x_n\}\} \\ &\leq d(x_n, T_n x_n) + \sup\left\{\sum_{k=n}^{\infty} d(T_k y, T_{k+1} y) : y \in \{x_n\}\right\} \\ &\leq d(x_n, T_n x_n) + \sum_{k=n}^{\infty} \sup\{d(T_k y, T_{k+1} y) : y \in \{x_n\}\}, \end{aligned}$$

which together with (3.5) and AKKT-condition imply

$$d(x_n, T x_n) \rightarrow 0. \quad (3.6)$$

Now, define $S_t x = tu \oplus (1-t)Tx$, for all $x \in C$. By Lemma 2.5, for all $t \in [0, 1]$, we suppose x_t is the unique fixed point of S_t . By Lemma 2.6, $\lim_{t \rightarrow 0} x_t = p$, where $p \in F(T)$ and p is the nearest point of $F(T)$ to u . In the sequel, we show $x_n \rightarrow p$. We have

$$\begin{aligned} d^2(x_{n+1}, p) &= d^2(\alpha_n u \oplus (1 - \alpha_n) y_n, p) \\ &\leq \alpha_n d^2(u, p) + (1 - \alpha_n) d^2(y_n, p) - \alpha_n (1 - \alpha_n) d^2(u, y_n) \\ &\leq \alpha_n d^2(u, p) + (1 - \alpha_n) (\beta_n d^2(x_n, p) + (1 - \beta_n) d^2(T_n z_n, p)) \\ &\quad - \alpha_n (1 - \alpha_n) d^2(u, y_n) \\ &\leq \alpha_n d^2(u, p) + (1 - \alpha_n) (\beta_n d^2(x_n, p) + (1 - \beta_n) d^2(z_n, p)) \\ &\quad - \alpha_n (1 - \alpha_n) d^2(u, y_n) \\ &\leq \alpha_n d^2(u, p) + (1 - \alpha_n) (\beta_n d^2(x_n, p) + (1 - \beta_n) (\gamma_n d^2(x_n, p) \\ &\quad + (1 - \gamma_n) d^2(T_n x_n, p))) - \alpha_n (1 - \alpha_n) d^2(u, y_n) \\ &\leq \alpha_n d^2(u, p) + (1 - \alpha_n) (\beta_n d^2(x_n, p) + (1 - \beta_n) d^2(x_n, p)) \\ &\quad - \alpha_n (1 - \alpha_n) d^2(u, y_n) \\ &= (1 - \alpha_n) d^2(x_n, p) + \alpha_n (d^2(u, p) - (1 - \alpha_n) d^2(u, y_n)), \end{aligned}$$

which implies

$$d^2(x_{n+1}, p) \leq (1 - \alpha_n)d^2(x_n, p) + \alpha_n(d^2(u, p) - (1 - \alpha_n)d^2(u, y_n)).$$

Hence, by A_1 and Lemma 2.4, it is enough to show

$$\limsup_n (d^2(u, p) - (1 - \alpha_n)d^2(u, y_n)) \leq 0.$$

For showing this, we have

$$\begin{aligned} d(y_n, x_n) &= d(\beta_n x_n \oplus (1 - \beta_n)T_n z_n, x_n) \\ &= (1 - \beta_n)d(T_n z_n, x_n) \\ &\leq (1 - \beta_n)d(T_n z_n, x_{n+1}) + (1 - \beta_n)d(x_{n+1}, x_n) \\ &\leq (1 - \beta_n)(\alpha_n d(u, T_n z_n) + (1 - \alpha_n)d(T_n z_n, y_n)) + (1 - \beta_n)d(x_{n+1}, x_n) \\ &= (1 - \beta_n)\alpha_n d(u, T_n z_n) + (1 - \beta_n)(1 - \alpha_n)\beta_n d(T_n z_n, x_n) + (1 - \beta_n)d(x_{n+1}, x_n) \end{aligned}$$

which follows

$$d(y_n, x_n) \rightarrow 0. \tag{3.7}$$

On the other hand, by (3.6) and Lemma 2.6, we get

$$\mu_n(d^2(u, p) - d^2(u, x_n)) \leq 0,$$

for all Banach limits μ . Thus

$$\limsup_n ((d^2(u, p) - d^2(u, x_{n+1})) - ((d^2(u, p) - d^2(u, x_n)))) = 0.$$

Therefore, by Lemma 2.3, we obtain

$$\limsup_n (d^2(u, p) - d^2(u, x_n)) \leq 0.$$

Hence, by (3.7) and $\alpha_n \rightarrow 0$, we get

$$\limsup_n (d^2(u, p) - (1 - \alpha_n)d^2(u, y_n)) = \limsup_n (d^2(u, p) - d^2(u, x_n)) \leq 0.$$

That is the desired result. □

The following corollary extends Theorem 2.1 of [15] as well as Theorem 3.1 of [6] and Theorem 2.3 of Saejung [17].

Corollary 3.2. *Let C be a closed and convex subset of complete CAT(0) space X and let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and the initial guess $x_0 \in C$ is chosen arbitrarily. Suppose $\{x_n\}$*

is generated by

$$\begin{cases} z_n = \gamma_n x_n \oplus (1 - \gamma_n)Tx_n, \\ y_n = \beta_n x_n \oplus (1 - \beta_n)Ty_n, \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)y_n, \end{cases} \quad (3.8)$$

where sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $[0, 1]$, satisfy the following conditions:

- (A₁) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (A₂) $\alpha_n \rightarrow 0$, $\beta_n \rightarrow 0$, $\liminf_n \gamma_n > 0$;
- (A₃) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$;
- (A₄) $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{|\beta_{n+1} - \beta_n|}{\alpha_{n+1}} = 0$;
- (A₅) $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{|\gamma_{n+1} - \gamma_n|}{\alpha_{n+1}} = 0$.

Then $\{x_n\}$ converges strongly to a fixed point of T , which is the nearest point of $F(T)$ to u .

In the following examples, we show the parallel conditions on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, respectively, in A_3, A_4 and A_5 are independent.

Example 3.3. Let $k \geq 1$ be a positive integer. If the sequence $\{\alpha_n\}$ is given by

$$\alpha_n = 0 \quad \text{for } n = k^2 \quad \text{and} \quad \alpha_n = \frac{1}{k^2} \quad \text{for } (k-1)^2 < n < k^2,$$

then $\alpha_n \rightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. But $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}}$ doesn't exist.

Example 3.4. Let $k \geq 0$ be an integer. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are given by

$$\alpha_n = \frac{1}{\sqrt{n+1}}, \beta_n = \frac{1}{k+1}, \text{ for } 2^k \leq n < 2^{k+1}, \gamma_n = \frac{k}{k+1} \text{ for } 2^k \leq n < 2^{k+1},$$

then $\alpha_n \rightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, $\beta_n \rightarrow 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\lim_{n \rightarrow \infty} \gamma_n = 1 > 0$ and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$. But $\lim_{n \rightarrow \infty} \frac{|\beta_{n+1} - \beta_n|}{\alpha_{n+1}} = \infty$ and $\lim_{n \rightarrow \infty} \frac{|\gamma_{n+1} - \gamma_n|}{\alpha_{n+1}}$ does not exist.

Example 3.5. If the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are given by

$$\alpha_n = \frac{1}{\sqrt{n+1}} \quad \text{and} \quad \beta_{2n} = \frac{1}{\sqrt{n+1}}, \beta_{2n+1} = \frac{1}{\sqrt{n+1}} + \frac{1}{n+1}$$

and $\gamma_{2n} = 1 - \frac{1}{n+1}$, $\gamma_{2n+1} = 1 - \frac{2}{n+1}$.

Then $\alpha_n \rightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, $\beta_n \rightarrow 0$, $\lim_{n \rightarrow \infty} \frac{|\beta_{n+1} - \beta_n|}{\alpha_{n+1}} = 0$, $\gamma_n \rightarrow 1$ and $\lim_{n \rightarrow \infty} \frac{|\gamma_{n+1} - \gamma_n|}{\alpha_{n+1}} = 0$. But $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| = \infty$ and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| = \infty$.

Example 3.6. Suppose that $\{\alpha_n\}$ is a sequence such that $\alpha_n \rightarrow 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. By setting $\beta_n = \alpha_n$ and $\gamma_n = 1 - \alpha_n$ for each n , we have
 (i) $\beta_n \rightarrow 0$ and $\gamma_n \rightarrow 1$,
 (ii) If $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, then $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.
 (iii) If $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, then $\lim_{n \rightarrow \infty} \frac{|\beta_{n+1} - \beta_n|}{\alpha_{n+1}} = 0$ and $\lim_{n \rightarrow \infty} \frac{|\gamma_{n+1} - \gamma_n|}{\alpha_{n+1}} = 0$.

In the following Corollary, with setting $T_n \equiv T$, we obtain an analogue of [23, Theorem 4.5] in CAT(0) spaces.

Corollary 3.7. *Let C be a closed and convex subset of complete CAT(0) space X and let $T : C \rightarrow C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and the initial guess $x_0 \in C$ is chosen arbitrarily. Assume that the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0, 1)$, satisfy in the following conditions:*
 (i) $a_n + b_n + c_n = 1, \forall n \geq 1$,
 (ii) $a_n \rightarrow 0, b_n \rightarrow 0$,
 (iii) $\sum_{n=1}^{\infty} a_n = \infty$,
 (iv) $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$, or $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0$,
 (v) $\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty$.
 Then $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= a_n u \oplus (1 - a_n) y_n, \\ y_n &= \frac{b_n}{1 - a_n} x_n \oplus \frac{c_n}{1 - a_n} T_n x_n, \end{aligned}$$

converges strongly to a fixed point of T which is the nearest point of $F(T)$ to u .

Proof. The proof is an immediate consequence of Theorem 2.1, by setting $\alpha_n = a_n, \beta_n = \frac{b_n}{1 - a_n}$ and $\gamma_n \equiv 1$. □

3.2 Inexact Halpern Iteration

In this short section, we consider the strong convergence of Halpern iteration with errors in the setting of CAT(0) spaces.

Theorem 3.8. *Let C be a closed and convex subset of complete CAT(0) space X and let $T : C \rightarrow C$ be a nonexpansive mapping with a nonempty fixed point set $F(T)$. Suppose that $u, x_1 \in C$ are arbitrary chosen and $\{x_n\}$ is iteratively*

generated by the following algorithm:

Step(1): Take $y_{n+1} = \alpha_n u \oplus (1 - \alpha_n)Tx_n$, $\forall n \geq 1$.

Step(2): Choose $x_{n+1} \in C$ such that $d(x_{n+1}, y_{n+1}) \leq e_n$, where $\{e_n\}$ is a sequence of nonnegative real numbers.

Step(3): Return to step(1).

Suppose $\{z_n\}$ be a sequence that generated by exact (without error) Halpern iteration

$$z_{n+1} = \alpha_n u \oplus (1 - \alpha_n)Tz_n, \quad \forall n \geq 1,$$

where $z_1 = x_1 \in C$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that

(1) $\lim_{n \rightarrow \infty} \alpha_n = 0$,

(2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$,

(4) $\sum_{n=1}^{\infty} e_n < \infty$ or $\lim_{n \rightarrow \infty} \frac{e_n}{\alpha_n} = 0$.

Then $\{x_n\}$ converges to $z \in F(T)$, which is the nearest point of $F(T)$ to u .

Proof. By [17, Theorem 2.3], $\{z_n\}$ converges to $z \in F(T)$ which is the nearest point of $F(T)$ to u . On the other hand,

$$\begin{aligned} d(z_{n+1}, x_{n+1}) &\leq d(z_{n+1}, y_{n+1}) + d(y_{n+1}, x_{n+1}) \\ &\leq (1 - \alpha_n)d(Tz_n, Tx_n) + e_n \\ &\leq (1 - \alpha_n)d(z_n, x_n) + e_n \end{aligned}$$

Thus, by the conditions (2), (4) and Lemma 2.4, we get

$$\lim_{n \rightarrow \infty} d(z_n, x_n) = 0. \tag{3.9}$$

On the other hand,

$$d(x_n, z) \leq d(x_n, z_n) + d(z_n, z),$$

which together with $z_n \rightarrow z$ and (3.9) imply x_n converges to $z \in F(T)$, which is the nearest point of $F(T)$ to u . □

Remark 3.9. Theorem 3.8 improves the sumability assumption on the error sequence $\{e_n\}$ even in Banach spaces. For example, $e_n = \frac{1}{n}$ and $\alpha_n = \frac{1}{\sqrt{n}}$ satisfy the assumptions of Theorem 3.8, but $\{e_n\}$ is not sumable.

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