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Positive bounded solutions for nonlinear polyharmonic problems in the unit ball

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Abstract

In this paper, we study the existence of positive solutions for the following nonlinear polyharmonic equation $(-\Delta)^m u+\lambda f(x,u)=0$ in B, subject to some boundary conditions, where m is a positive integer, λ is a nonnegative constant and B is the unit ball of $\mathbb{R}^n$ $(n\geq 2)$. Under some appropriate assumptions on the nonnegative nonlinearity term $f(x,u)$ and by using the Schäuder fixed point theorem, the existence of positive solutions is obtained. At last, examples are given for illustration.

1 Introduction

The goal of the paper is to study the existence of positive continuous bounded solutions for the following nonlinear elliptic higher order problem :

$$\begin{cases} (-\Delta)^m u+\lambda f(x,u)=0 \text{ in } B, \\ u>0 \text{ in } B, \\ \lim\limits_{x\to\xi\in\partial B}\frac{u(x)}{(1-|x|^2)^{m-1}}=\varphi(\xi), \end{cases} \tag{1.1}$$

where m is a positive integer, $B=\{x\in\mathbb{R}^n : |x|<1\}$ is the unit ball of $\mathbb{R}^n$ $(n\geq 2)$, $\partial B=\{x\in\mathbb{R}^n : |x|=1\}$ is the boundary of B, λ is a nonnegative constant, φ is a nontrivial nonnegative continuous function on ∂B and $f : B\times[0,\infty)\to[0,\infty)$ is continuous.

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The polyharmonic operator $(-\Delta)^m$, $m \in \mathbb{N}^*$, has been studied several years later. Indeed, in [5], Boggio showed that the Green function $G_{m,n}$ of the operator $(-\Delta)^m$ on B with Dirichlet boundary conditions $u = \frac{\partial}{\partial \nu} u = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0$ on ∂B, is given by :

$$G_{m,n}(x,y) = k_{m,n} \left| x - y \right|^{2m-n} \int_1^{\frac{[x,y]}{|x-y|}} \frac{(\nu^2 - 1)^{m-1}}{\nu^{n-1}} d\nu,$$

where $k_{m,n}$ is a positive constant, $\frac{\partial}{\partial \nu}$ is the outward normal derivative and for x, y in B, $[x,y]^2 = |x-y|^2 + \left(1 - |x|^2\right)\left(1 - |y|^2\right)$.
In [2], the estimates on the Green function $G_{m,n}$ of $(-\Delta)^m$ on B and particulary the $3G-$theorem (see [2], Theorem 2.8), allowed the authors to introduce a large functional class called Kato class denoted by $K_{m,n}$ (see Definition 1 below). This class plays a key role in the study of some nonlinear polyharmonic equations (see [2, 3, 6, 10]). For related results we refer to the recent monograph [12] and the papers [7, 9, 11, 13, 14, 15].

Definition 1. (*See* [2]) *A Borel measurable function q on B belongs to the Kato class $K_{m,n}$ if q satisfies the following condition :*

$$\lim_{\alpha \to 0} \left(\sup_{x \in B} \int_{B \cap B(x,\alpha)} \left(\frac{\delta(y)}{\delta(x)} \right)^m G_{m,n}(x,y) \left| q(y) \right| dy \right) = 0.$$

Here and always $\delta(x) = 1 - |x|$ is the Euclidian distance between x and ∂B.

As typical example of functions belonging to the class $K_{m,n}$, we quote

Example 1. (*See* [3]) *The function q defined on B by*

$$q(x) = \frac{1}{(\delta(x))^{\lambda} (\log \frac{2}{\delta(x)})^{\mu}},$$

is in $K_{m,n}$ if and only if $\lambda < 2m$ and $\mu \in \mathbb{R}$ or $\lambda = 2m$ and $\mu > 1$.

Before presenting our main result, we lay out a number of potential theory tools and some notations which will be used throughout the paper. Let φ be a nontrivial nonnegative continuous function on ∂B, we denote by $H\varphi$ the bounded continuous solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 \text{ in } B, \\ u_{/\partial B} = \varphi. \end{cases}$$

We set ω the function defined on B by

$$\omega : x \to (1 - |x|^2)^{m-1} H\varphi(x).$$

We remark that the function ω is a bounded continuous solution of the problem

$$\begin{cases} (-\Delta)^m \omega = 0 \quad \text{in } B, \\ \lim\limits_{x \to \xi \in \partial B} \frac{\omega(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi). \end{cases} \tag{1.2}$$

For simplicity, we denote by

$$C_0(B) = \{v \text{ continuous on } B \text{ and } \lim_{x \to \xi \in \partial B} v(x) = 0\}.$$

We also refer to $V_{m,n}g$ the $m-$potentiel of a nonnegative measurable function g on B by

$$V_{m,n}g(x) = \int_B G_{m,n}(x,y)\, g(y)dy, \ \ x \in B.$$

Recall that for each nonnegative measurable function g on B such that g and $V_{m,n}g$ are in $L^1_{loc}(B)$, we have

$$(-\Delta)^m (V_{m,n}g) = g \ \ \text{(in the distributional sense)}.$$

We assume that the function f satisfies the following assumptions :

($\mathbf{H}_1$) $f : B \times [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with respect to the second variable.

($\mathbf{H}_2$) The function $q = \dfrac{f(.,\omega)}{\omega}$ belongs to the Kato class $K_{m,n}$.

Theorem 1. *Assume* $(H_1) - (H_2)$. *Then there exists* $\lambda_0 > 0$ *such that for each* $\lambda \in [\,0, \lambda_0)$, *problem* (1.1) *has a positive continuous solution* u *satisfying for each* $x \in B$

$$\left(1 - \frac{\lambda}{\lambda_0}\right) \omega(x) \le u\,(x) \le \omega(x). \tag{1.3}$$

We remark that for $m = 1$, we find again the result of [1] which was our original motivation for deriving our study.

Remark 1. *Note that problem (1.1) is a perturbation of problem (1.2). In view of (1.3), we see that the behavior of the obtained solution is not affected by the perturbation term.*

The outline of this paper is as follows. In Section 2, we state some already known results on the Green function $G_{m,n}$ and the functions in the class $K_{m,n}$ that will be used in our study. Section 3 is devoted to the proof of Theorem 1. The last section is reserved for some examples.

Finally, we mention that the letter c will be a positive generic constant which may vary from line to line.

2 Properties of the Green function $G_{m,n}$ and the class $K_{m,n}$

To make the paper self contained, this section is devoted to recall some results established in [2, 4, 8, 10] that will be useful for our study.

Proposition 1. *If $x, y \in B$ such that $|x-y| \geq r > 0$, then there exists $c > 0$ such that*

$$G_{m,n}(x,y) \leq c\frac{(\delta(x)\delta(y))^m}{r^n}.$$

Proposition 2. *Let q be a function in $K_{m,n}$, then*

(i) The constant $\alpha_q = \sup_{x,y\in B} \int_B \frac{G_{m,n}(x,z)G_{m,n}(z,y)}{G_{m,n}(x,y)} |q(z)|\, dz$ is finite.

(ii) The function $x \mapsto (\delta(x))^{2m-1}q(x)$ is in $L^1(B)$.

(iii) For each nonnegative harmonic function h in B, we have for $x \in B$

$$\int_B G_{m,n}(x,y)(1-|y|^2)^{m-1}h(y)\,|q(y)|\,dy \leq \alpha_q(1-|x|^2)^{m-1}h(x).$$

(iv) For each $x_0 \in \overline{B}$, we have

$$\lim_{\alpha\to 0}\left(\sup_{x\in B}\int_{B\cap B(x_0,\alpha)}\left(\frac{\delta(y)}{\delta(x)}\right)^{m-1} G_{m,n}(x,y)\,|q(y)|\,dy\right) = 0.$$

(v) The function $x \mapsto \int_B \left(\frac{\delta(y)}{\delta(x)}\right)^{m-1} G_{m,n}(x,y)\,|q(y)|\,dy$ is in $C_0(B)$.

3 Proof of Theorem 1

We begin this section with the following lemma which plays a key role in the proof of Theorem 1.

Lemma 1. *If f satisfies (H_2), then*

$$\lambda_0 := \inf_{x\in B}\frac{\omega(x)}{V_{m,n}(f(.,\omega))(x)} > 0. \tag{3.1}$$

Proof. Since f satisfies (H_2), then the function $q = \frac{f(.,\omega)}{\omega}$ belongs to the Kato class $K_{m,n}$. It follows from Proposition 2 (iii) that for each $x \in B$,

$$V_{m,n}(f(.,\omega))(x) = V_{m,n}(q\,\omega)(x) \leq \alpha_q\,\omega(x).$$

This gives that for each $x \in B$,

$$\frac{\omega(x)}{V_{m,n}(f(.,\omega))(x)} \geq \frac{1}{\alpha_q}.$$

This implies that

$$\lambda_0 \geq \frac{1}{\alpha_q} > 0.$$

□

Now, we are ready to prove our main result.

Proof of Theorem 1. Let Λ be the non-empty closed convex set given by

$$\Lambda = \{v \in C_0(B) : \left(1 - \frac{\lambda}{\lambda_0}\right)\omega \leq v \leq \omega\}.$$

We define the operator T on Λ by

$$Tv = \omega - \lambda V_{m,n}\left(f(.,v)\right).$$

We aim to prove that T has a fixed point in Λ. First, we shall prove that $T\Lambda$ is relatively compact in $C_0(B)$. Since $\omega \in C_0(B)$, it is enough to show that the family

$$\{V_{m,n}\left(f(.,v)\right), v \in \Lambda\}$$

is relatively compact in $C_0(B)$.
Let $v \in \Lambda$, then by hypothesis (H_1) we obtain that

$$0 \leq V_{m,n}\left(f(.,v)\right) \leq V_{m,n}\left(f(.,\omega)\right) = V_{m,n}(q\,\omega). \tag{3.2}$$

Applying Proposition 2 (iii), we get

$$0 \leq V_{m,n}\left(f(.,v)\right) \leq \alpha_q \left\|\omega\right\|_\infty.$$

Thus the family $\{V_{m,n}\left(f(.,v)\right),\ v \in \Lambda\}$ is uniformly bounded.
Now, we aim at proving that $\{V_{m,n}\left(f(.,v)\right),\ v \in \Lambda\}$ is equicontinuous on B. Let $x_0 \in B$ and $\varepsilon > 0$. By Proposition 2 (iv), there exists $\alpha > 0$ such that

$$0 \leq \sup_{z \in B} \int_{B \cap B(x_0, 2\alpha)} \left(\frac{\delta(y)}{\delta(\xi)}\right)^{m-1} G_{m,n}(z,y)q(y)dy \leq \frac{\varepsilon}{2^m \left\|H\varphi\right\|_\infty}. \tag{3.3}$$

Let $x, x' \in B \cap B(x_0, \alpha)$, then for each $v \in \Lambda$, we have

$$\begin{aligned}
& \left|V_{m,n}\left(f(.,v)\right)(x) - V_{m,n}\left(f(.,v)\right)(x')\right| \\
\leq \quad & \int_B \left|G_{m,n}(x,y) - G_{m,n}(x',y)\right| q(y)\omega(y)dy \\
\leq \quad & 2^{m-1}\left\|H\varphi\right\|_\infty \int_B \left|G_{m,n}(x,y) - G_{m,n}(x',y)\right| \left(\delta(y)\right)^{m-1} q(y)dy \\
\leq \quad & 2^{m-1}\left\|H\varphi\right\|_\infty \int_{B\cap B(x_0,2\alpha)} \left|G_{m,n}(x,y) - G_{m,n}(x',y)\right| \left(\delta(y)\right)^{m-1} q(y)dy \\
& +2^{m-1}\left\|H\varphi\right\|_\infty \int_{B\cap B^c(x_0,2\alpha)} \left|G_{m,n}(x,y) - G_{m,n}(x',y)\right| \left(\delta(y)\right)^{m-1} q(y)dy \\
:= \quad & I_1 + I_2.
\end{aligned}$$

From (3.3), we get that

$$\begin{aligned}
I_1 \quad \leq \quad & 2^{m-1}\left\|H\varphi\right\|_\infty \int_{B\cap B(x_0,2\alpha)} \left(\frac{\delta(y)}{\delta(x)}\right)^{m-1} G_{m,n}(x,y)q(y)dy \\
& +2^{m-1}\left\|H\varphi\right\|_\infty \int_{B\cap B(x_0,2\alpha)} \left(\frac{\delta(y)}{\delta(x')}\right)^{m-1} G_{m,n}(x',y)q(y)dy \\
\leq \quad & 2^{m}\left\|H\varphi\right\|_\infty \sup_{z\in B} \int_{B\cap B(x_0,2\alpha)} \left(\frac{\delta(y)}{\delta(z)}\right)^{m-1} G_{m,n}(z,y)q(y)dy \\
\leq \quad & \varepsilon
\end{aligned}$$

On the other hand, if $|y - x_0| \geq 2\alpha$ then $|y - x| \geq \alpha$ and $|y - x'| \geq \alpha$. So by applying Proposition 1, we reach

$$\begin{aligned}
& \left|G_{m,n}(x,y) - G_{m,n}(x',y)\right| \left(\delta(y)\right)^{m-1} q(y) \\
& \quad \leq \left(G_{m,n}(x,y) + G_{m,n}(x',y)\right)(\delta(y))^{m-1}q(y) \\
& \quad \leq \frac{c}{\alpha^n}\left((\delta(x)\delta(y))^m + (\delta(x')\delta(y))^m\right)(\delta(y))^{m-1}q(y) \\
& \quad \leq c(\delta(y))^{2m-1}q(y).
\end{aligned}$$

Now, since for $y \in B\cap B^c(x_0, 2\alpha)$, $x \mapsto G_{m,n}(x,y)$ is continuous in $B\cap B(x_0,\alpha)$ and from Proposition 2 (ii) the function $y \mapsto \left(\delta(y)\right)^{2m-1} q(y)$ is in $L^1(B)$ then we deduce by the dominated convergence theorem that

$$I_2 \to 0 \text{ as } |x - x'| \to 0.$$

Thus $\left\{V_{m,n}\left(f(.,v)\right), v \in \Lambda\right\}$ is equicontinuous on B.

Next, we claim that $V_{m,n}(f(.,v))(x) \to 0$ as $x \to \xi \in \partial B$ uniformly in $v \in \Lambda$. Let $\xi \in \partial B$ and $x \in B \cap B(\xi, \alpha)$. Then for each $v \in \Lambda$, we have from (3.2)

$$\begin{aligned} V_{m,n}(f(.,v))(x) &\leq 2^{m-1}\|H\varphi\|_\infty \int_B G_{m,n}(x,y)(\delta(y))^{m-1} q(y)dy \\ &\leq 2^{m-1}\|H\varphi\|_\infty \int_{B\cap B(\xi,2\alpha)} G_{m,n}(x,y)(\delta(y))^{m-1} q(y)dy \\ &\quad +2^{m-1}\|H\varphi\|_\infty \int_{B\cap B^c(\xi,2\alpha)} G_{m,n}(x,y)(\delta(y))^{m-1} q(y)dy \\ &\leq 2^{m-1}\|H\varphi\|_\infty \sup_{z\in B}\int_{B\cap B(\xi,2\alpha)} \left(\frac{\delta(y)}{\delta(z)}\right)^{m-1} G_{m,n}(z,y)q(y)dy \\ &\quad +2^{m-1}\|H\varphi\|_\infty \int_{B\cap B^c(\xi,2\alpha)} G_{m,n}(x,y)(\delta(y))^{m-1} q(y)dy. \end{aligned}$$

By applying Proposition 2 (iv), we obtain that

$$2^{m-1}\|H\varphi\|_\infty \sup_{z\in B}\int_{B\cap B(\xi,2\alpha)} \left(\frac{\delta(y)}{\delta(z)}\right)^{m-1} G_{m,n}(z,y)q(y)dy \to 0 \text{ as } \alpha \to 0.$$

For $y \in B \cap B^c(\xi, 2\alpha)$, we have $|x-y| \geq \alpha$. Hence, it follows from Proposition 1 and Proposition 2 (ii) that

$$\begin{aligned} &2^{m-1}\|H\varphi\|_\infty \int_{B\cap B^c(\xi,2\alpha)} G_{m,n}(x,y)(\delta(y))^{m-1} q(y)dy \\ \leq\ & c\left(\int_B (\delta(y))^{2m-1} q(y)dy\right)(\delta(x))^m \to 0 \text{ as } x \to \xi. \end{aligned}$$

Therefore by Ascoli's theorem, we conclude that the family $\{V_{m,n}(f(.,v)),\ v \in \Lambda\}$ is relatively compact in $C_0(B)$.
For $v \in \Lambda$, we have from (3.2)

$$\omega - \lambda V_{m,n}(f(.,\omega)) \leq Tv \leq \omega.$$

This implies from (3.1) that

$$\left(1 - \frac{\lambda}{\lambda_0}\right)\omega \leq Tv \leq \omega.$$

Combining this with the fact that $Tv \in C_0(B)$, we deduce that $T\Lambda \subset \Lambda$.

Now, we prove the continuity of T in Λ in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges uniformly to a function v in Λ. Then for $k \in \mathbb{N}$ and each $x \in B$,

$$|Tv_k(x) - Tv(x)| \leq \int_B G_{m,n}(x,y)\,|f(y,v_k(y)) - f(y,v(y))|\,dy.$$

On the other hand, from the monotonicity of the function f, we have for $k \in \mathbb{N}$ and $(x, y) \in B^2$,

$$\begin{aligned} G_{m,n}(x,y)\left|f(y,v_k(y)) - f(y,v(y))\right| &\leq 2G_{m,n}(x,y)f(y,\omega(y)) \\ &\leq 2G_{m,n}(x,y)q(y)\omega(y). \end{aligned}$$

Since by Proposition 2 (iii), $\int_B G_{m,n}(x,y)q(y)\omega(y)dy \leq \alpha_q \left\|\omega\right\|_\infty < \infty$, we conclude by the continuity of f with respect to the second variable and the dominated convergence theorem that

$$|Tv_k(x) - Tv(x)| \to 0 \text{ as } k \to \infty.$$

Consequently, since $T\Lambda$ is relatively compact in $C_0(B)$, we deduce that the pointwise convergence implies the uniform convergence, namely

$$\left\|Tv_k - Tv\right\|_\infty \to 0 \text{ as } k \to \infty.$$

Thus we have proved that T is a compact mapping from Λ to itself. Hence the Schäuder's fixed point theorem implies the existence of $u \in \Lambda$ such that

$$u = Tu,$$

that is

$$u = \omega - \lambda V_{m,n}\left(f(.,u)\right). \tag{3.4}$$

It is clear that u is continuous and satisfies (1.3) and it remains to verify that u is a solution of (1.1).
Since $0 \leq f(.,u) \leq 2^{m-1}\left\|H\varphi\right\|_\infty (\delta(.))^{m-1} q$ then by Proposition 2 (ii), we obtain that $f(.,u) \in L^1_{loc}(B)$ and from (3.4), we have $V_{m,n}\left(f(.,u)\right) \in L^1_{loc}(B)$. Hence, we have in the distributional sense

$$(-\Delta)^m V_{m,n}\left(f(.,u)\right) = f(.,u) \text{ in } B.$$

Now, applying $(-\Delta)^m$ on both sides of (3.4), we obtain that

$$(-\Delta)^m u = -\lambda f(.,u) \text{ in } B.$$

Finally, we have

$$\lim_{x\to\xi\in\partial B}\frac{u(x)}{(1-|x|^2)^{m-1}} = \varphi(\xi) - \lambda \lim_{x\to\xi\in\partial B}\frac{V_{m,n}\left(f(.,u)\right)(x)}{(1-|x|^2)^{m-1}}.$$

Since for $x \in B$, we have

$$0 \leq \frac{V_{m,n}\left(f(.,u)\right)(x)}{(1-|x^2|)^{m-1}} \leq 2^{m-1}\left\|H\varphi\right\|_\infty \int_B \left(\frac{\delta(y)}{\delta(x)}\right)^{m-1} G_{m,n}(x,y)q(y)dy,$$

we deduce by Proposition 2 (v) that

$$\lim_{x\to\xi\in\partial B}\frac{V_{m,n}\left(f(.,u)\right)(x)}{(1-|x|^2)^{m-1}}=0.$$

Hence

$$\lim_{x\to\xi\in\partial B}\frac{u(x)}{(1-|x|^2)^{m-1}}=\varphi(\xi).$$

This ends the proof. □

4 Examples

We take up in this section some examples illustrating our main result.

Example 2. *Let φ be a positive continuous function on ∂B. Let p be a nonnegative measurable function satisfying for each $x\in B$, $p(x)\leq\frac{c}{(\delta(x))^{\lambda}(\log(\frac{2}{\delta(x)}))^{\mu}}$ with $\lambda<2m$ and $\mu\in\mathbb{R}$ or $\lambda=2m$ and $\mu>1$. We consider $g:[0,\infty)\to[0,\infty)$ nondecreasing and continuous function such that for each $c>0$, there exists $\eta>0$ satisfying*

$$g(t)\leq\eta t,\quad\forall t\in[0,c]\,.$$

Then by Theorem 1, there exists $\lambda_0>0$ such that for each $\lambda\in[0,\lambda_0)$, the problem

$$\left\{\begin{array}{l}(-\Delta)^m u+\lambda p(x)g(u)=0 \ \text{in } B,\\ u>0 \ \text{in } B,\\ \lim\limits_{x\to\xi\in\partial B}\frac{u(x)}{(1-|x|^2)^{m-1}}=\varphi(\xi),\end{array}\right.$$

has a positive continuous solution u satisfying (1.3).

Example 3. *Let φ be a positive continuous function on ∂B. Let p be a nonnegative measurable function satisfying for each $x\in B$, $p(x)\leq\frac{1}{(\delta(x))^{\lambda}}$ with $\lambda<2m$ and f be the function defined on $B\times[0,\infty)$ by $f(x,u)=p(x)u^{\sigma}$ with $\sigma\geq1$. Therefore by Theorem 1, there exists $\lambda_0>0$ such that for each $\lambda\in[0,\lambda_0)$, the problem*

$$\left\{\begin{array}{l}(-\Delta)^m u+\lambda p(x)u^{\sigma}=0 \ \text{in } B,\\ u>0 \ \text{in } B,\\ \lim\limits_{x\to\xi\in\partial B}\frac{u(x)}{(1-|x|^2)^{m-1}}=\varphi(\xi),\end{array}\right.$$

has a positive continuous solution u satisfying (1.3).

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