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TOPOLOGICAL TRANSVERSALITY PRINCIPLES AND GENERAL COINCIDENCE THEORY

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Abstract

This paper presents general topological coincidence principles for multivalued maps defined on subsets of completely regular topological spaces.

1. Introduction.

The notion of an essential map was introduced by Granas [2] and extended in a variety of setting; see [1, 5, 6, 7] and the references therein. In this paper we present a general continuation theory for coincidences. Our theory relies on a Urysohn type lemma and on the notion of d–Φ–essential and d–L–Φ–essential maps. In particular we present a general topological transversality type theorem which extends results in the literature; see [1, 3, 4, 6, 7] and the references therein.

2. d–Φ–essential maps.

Let E be a completely regular topological space and U an open subset of E.

We will consider classes $\mathbf{A}$ and $\mathbf{B}$ of maps.

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Definition 2.1. We say $F \in A(\overline{U}, E)$ (respectively $F \in B(\overline{U}, E)$) if $F : \overline{U} \to 2^E$ and $F \in \mathbf{A}(\overline{U}, E)$ (respectively $F \in \mathbf{B}(\overline{U}, E)$); here 2^E denotes the family of nonempty subsets of E.

In this section we <u>fix</u> a $\Phi \in B(\overline{U}, E)$.

Definition 2.2. We say $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

For any map $F \in A(\overline{U}, E)$ let $F^\star = I \times F : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}\ (B) \right\} \cup \{\emptyset\} \to \Omega \tag{2.1}$$

be any map with values in the nonempty set Ω; here $B = \left\{ (x, \Phi(x)) :\ x \in \overline{U} \right\}$.

Definition 2.3. Let E be a completely regular (respectively normal) topological space, and U an open subset of E. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and $\left\{ x \in \overline{U} :\ (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1] \right\}$ is compact (respectively closed); here $H^\star(x,t) = (x\,,\, H(x,t))$ and $H_t(x) = H(x,t)$.

The following conditions will be assumed:

$$\cong \quad \text{is an equivalence relation in } A_{\partial U}(\overline{U}, E), \tag{2.2}$$

and

$$\left\{ \begin{array}{l} \text{if } F, G \in A_{\partial U}(\overline{U}, E) \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \cong G \\ \text{in } A_{\partial U}(\overline{U}, E) \text{ then } d\left((F^\star)^{-1}\ (B) \right) = d\left((G^\star)^{-1}\ (B) \right). \end{array} \right. \tag{2.3}$$

Definition 2.4. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d–Φ–essential if $d\left((F^\star)^{-1}\ (B) \right) \neq d(\emptyset)$. We say $F^\star$ is d–Φ–inessential if $d\left((F^\star)^{-1}\ (B) \right) = d(\emptyset)$.

Remark 2.1. If $F^\star$ is d–Φ–essential then

$$\begin{aligned} \emptyset \neq (F^\star)^{-1}\ (B) &= \{x \in \overline{U} :\ F^\star(x) \cap B \neq \emptyset\} \\ &= \{x \in \overline{U} :\ (x, F(x)) \cap (x, \Phi(x)) \neq \emptyset\}, \end{aligned}$$

and this together with $F(x) \cap \Phi(x) = \emptyset$ for $x \in \partial U$ implies that there exists $x \in U$ with $(x, \Phi(x)) \cap F^\star(x) \neq \emptyset$ (i.e. $\Phi(x) \cap F(x) \neq \emptyset$).

Theorem 2.1. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d a map defined in (2.1) and assume (2.2) and (2.3) hold. Suppose $F \in A_{\partial U}(\overline{U}, E)$ and assume the following condition holds:*

$$(2.4) \quad \begin{cases} \text{if there exists a map } G \in A_{\partial U}(\overline{U}, E) \text{ with } G \cong F \text{ in} \\ A_{\partial U}(\overline{U}, E) \text{ and } d\left((G^\star)^{-1}(B)\right) = d(\emptyset) \text{ with } G^\star = I \times G, \\ \text{and if } H \text{ is the map defined in Definition 2.3 and} \\ \mu : \overline{U} \to [0,1] \text{ is any continuous map with } \mu(\partial U) = 0, \text{ then} \\ \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, t\,\mu(x))) \neq \emptyset \text{ for some } t \in [0,1]\right\} \\ \text{is closed.} \end{cases}$$

Then the following are equivalent:

(i). $F^\star = I \times F : \overline{U} \to 2^{\overline{U} \times E}$ is d–Φ–inessential;

(ii). there exists a map $G \in A_{\partial U}(\overline{U}, E)$ with $G^\star = I \times G$ and $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((G^\star)^{-1}(B)\right) = d(\emptyset)$.

PROOF: (i) implies (ii) is immediate. Next we prove (ii) implies (i). Suppose there exists a map $G \in A_{\partial U}(\overline{U}, E)$ with $G^\star = I \times G$ and $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((G^\star)^{-1}(B)\right) = d(\emptyset)$. Let $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = G$, $H_0 = F$ and $\left\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$ is compact (respectively closed); here $H^\star(x,t) = (x\,,\, H(x,t))$ and $H_t(x) = H(x,t)$). Consider

$$D = \left\{x \in \overline{U} : (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

If $D = \emptyset$ then in particular $\emptyset = (x, \Phi(x)) \cap H^\star(x, 0) = (x, \Phi(x)) \cap F^\star(x)$ for $x \in \overline{U}$ i.e. $(F^\star)^{-1}(B) = \emptyset$ so $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ i.e. $F^\star$ is d–Φ–inessential. Next suppose $D \neq \emptyset$. Note D is compact (respectively closed). Also $D \cap \partial U = \emptyset$ since $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to 2^E$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x)$ and let $R_\mu^\star = I \times R_\mu$. Note $R_\mu \in A(\overline{U}, E)$, $R_\mu|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ since $\mu(\partial U) = 0$, and $R_\mu \in A_{\partial U}(\overline{U}, E)$.

Also note since $\mu(D) = 1$ that

$$\begin{aligned} \left(R_\mu^\star\right)^{-1}(B) &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, \mu(x)) \neq \emptyset\right\} \\ &= \left\{x \in \overline{U} : (x, \Phi(x)) \cap (x, H(x, 1) \neq \emptyset\right\} = (G^\star)^{-1}(B) \end{aligned}$$

so

$$d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) = d(\emptyset). \tag{2.5}$$

We claim

$$R_\mu \cong F \ \text{ in } \ A_{\partial U}(\overline{U}, E). \tag{2.6}$$

If (2.6) is true then (2.3) and (2.5) guarantee that

$$d\left((F^\star)^{-1}(B)\right) = d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d(\emptyset),$$

so $F^\star$ is d–Φ–inessential.

It remains to show (2.6). Let $Q : \overline{U} \times [0,1] \to 2^E$ be given by $Q(x,t) = H(x, t\,\mu(x))$. Note $Q(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$ and (see (2.4) and Definition 2.3)

$$\begin{aligned}\{x \in \overline{U} :\ \emptyset \ &\neq\ (x, \Phi(x)) \cap (x, Q(x,t)) \\ &=\ (x, \Phi(x)) \cap (x, H(x, t\,\mu(x))) \ \text{ for some } \ t \in [0,1]\}\end{aligned}$$

is compact (respectively closed). Note $Q_0 = F$ and $Q_1 = R_\mu$. Finally if there exists a $t \in [0,1]$ and $x \in \partial U$ with $\Phi(x) \cap Q_t(x) \neq \emptyset$ then $\Phi(x) \cap H_{t\,\mu(x)}(x) \neq \emptyset$, so $x \in D$ and so $\mu(x) = 1$ i.e. $\Phi(x) \cap H_t(x) \neq \emptyset$, a contradiction. Thus (2.6) holds. □

Now Theorem 2.1 immediately yields the following continuation theorem.

Theorem 2.2. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) :\ x \in \overline{U}\}$, d a map defined in* (2.1) *and assume* (2.2), (2.3) *and* (2.4) *hold. Suppose J and Ψ are two maps in $A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $\Psi^\star = I \times \Psi$ and with $J \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$. Then $J^\star$ is d–Φ–inessential if and only if $\Psi^\star$ is d–Φ–inessential.*

PROOF: Assume $J^\star$ is d–Φ–inessential. Then (see Theorem 2.1) there exists a map $Q \in A_{\partial U}(\overline{U}, E)$ with $Q^\star = I \times Q$ and $Q \cong J$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((Q^\star)^{-1}(B)\right) = d(\emptyset)$. Note (since $\cong$ is an equivalence relation in $A_{\partial U}(\overline{U}, E)$) also that $Q \cong \Psi$ in $A_{\partial U}(\overline{U}, E)$. Then Theorem 2.1 (with $F = \Psi$ and $G = Q$) guarantees that $\Psi^\star$ is d–Φ–inessential. Similarly if $\Psi^\star$ is d–Φ–inessential then $J^\star$ is d–Φ–inessential. □

We now show how the ideas in this section can be applied to other natural situations. Let E be a Hausdorff topological vector space (so automatically a completely regular space), Y a topological vector space, and U an open subset of E. Also let $L : dom\,L \subseteq E \to Y$ be a linear single valued map;

here $dom\, L$ is a vector subspace of E. Finally $T : E \to Y$ will be a linear single valued map with $L + T : dom\, L \to Y$ a bijection; for convenience we say $T \in H_L(E, Y)$.

Definition 2.5. We say $F \in A(\overline{U}, Y; L, T)$ (respectively $F \in B(\overline{U}, Y; L, T)$) if $F : \overline{U} \to 2^Y$ and $(L+T)^{-1}(F+T) \in A(\overline{U}, E)$ (respectively $(L+T)^{-1}(F+T) \in B(\overline{U}, E)$).

We now fix a $\Phi \in B(\overline{U}, Y; L, T)$.

Definition 2.6. We say $F \in A_{\partial U}(\overline{U}, Y; L, T)$ if $F \in A(\overline{U}, Y; L, T)$ with $(L+T)^{-1}(F+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for $x \in \partial U$.

For any map $F \in A(\overline{U}, Y; L, T)$ let $F^\star = I \times (L+T)^{-1}(F+T) : \overline{U} \to 2^{\overline{U} \times E}$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1}\ (B) \right\} \cup \{\emptyset\} \to \Omega \tag{2.7}$$

be any map with values in the nonempty set Ω; here

$$B = \left\{ (x, (L+T)^{-1}(\Phi+T)(x)) : \ x \in \overline{U} \right\}.$$

Definition 2.7. Let $F, G \in A_{\partial U}(\overline{U}, Y; L, T)$. Now $F \cong G$ in $A_{\partial U}(\overline{U}, Y; L, T)$ if there exists a map $H : \overline{U} \times [0,1] \to 2^Y$ with $(L+T)^{-1}(H(\,.\,, \eta(\,.\,)) + T(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t+T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = F$, $H_0 = G$ and

$$\left\{ x \in \overline{U} : \ (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1] \right\}$$

is compact; here $H_t(x) = H(x,t)$ and $H^\star(x,\lambda) = (x\,,\ (L+T)^{-1}(H+T)(x,\lambda))$.

The following conditions will be assumed:

$$\cong \ \text{ is an equivalence relation in } \ A_{\partial U}(\overline{U}, Y; L, T), \tag{2.8}$$

and

$$\begin{cases} \text{if } \ F, G \in A_{\partial U}(\overline{U}, Y; L, T) \ \text{ with } F|_{\partial U} = G|_{\partial U} \text{ and } F \cong G \\ \text{in } \ A_{\partial U}(\overline{U}, Y; L, T) \ \text{ then } d\left((F^\star)^{-1}\ (B) \right) = d\left((G^\star)^{-1}\ (B) \right). \end{cases} \tag{2.9}$$

Definition 2.8. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d–L–Φ–essential if $d\left((F^\star)^{-1}\ (B) \right) \neq d(\emptyset)$. We say $F^\star$ is d–L–Φ–inessential if $d\left((F^\star)^{-1}\ (B) \right) = d(\emptyset)$.

Theorem 2.3. *Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $B = \{(x, (L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U}\}$, $L : dom\, L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, d a map defined in (2.7) and assume (2.8) and (2.9) hold. Suppose $F \in A_{\partial U}(\overline{U}, Y; L, T)$ and assume the following condition holds:*

$$(2.10)\qquad \begin{cases} \textit{if there exists a map } G \in A_{\partial U}(\overline{U}, Y; L, T) \textit{ with } G \cong F \\ \textit{in } A_{\partial U}(\overline{U}, Y; L, T) \textit{ and } d\left((G^\star)^{-1}(B)\right) = d(\emptyset) \textit{ with} \\ G^\star = I \times (L+T)^{-1}(G+T) \textit{ and if } H \textit{ is the map} \\ \textit{defined in Definition 2.7 and } \mu : \overline{U} \to [0,1] \textit{ is any} \\ \textit{continuous map with } \mu(\partial U) = 0, \textit{ then} \\ \{x \in \overline{U} : \emptyset \neq (x, (L+T)^{-1}(\Phi+T)(x)) \\ \quad \cap\, (x, (L+T)^{-1}(H_{t\,\mu(x)} + T)(x)) \textit{ for some } t \in [0,1]\} \\ \textit{is closed.} \end{cases}$$

Then the following are equivalent:

(i). $F^\star = I \times (L+T)^{-1}(F+T) : \overline{U} \to 2^{\overline{U}\times E}$ is d–L–Φ–inessential;

(ii). there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G^\star = I \times (L+T)^{-1}(G+T)$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((G^\star)^{-1}(B)\right) = d(\emptyset)$.

PROOF: (i) implies (ii) is immediate. Next we prove (ii) implies (i). Suppose there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G^\star = I \times (L+T)^{-1}(G+T)$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((G^\star)^{-1}(B)\right) = d(\emptyset)$. Let $H : \overline{U} \times [0,1] \to 2^Y$ be a map with $(L+T)^{-1}(H(\,.\,, \eta(\,.\,)) + T(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}(H_t + T)(x) \cap (L+T)^{-1}(\Phi+T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = G$, $H_0 = F$ (here $H_t(x) = H(x,t)$) and

$$\left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact; here $H^\star(x,\lambda) = (x\,,\, (L+T)^{-1}(H+T)(x,\lambda))$.

Let

$$D = \left\{x \in \overline{U} : (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

If $D = \emptyset$ then in particular ($H_0 = F$) note $\emptyset = (x, (L+T)^{-1}(\Phi+T)(x)) \cap (x, (L+T)^{-1}(F+T)(x))$, so $F^\star$ in d–L–Φ–inessential. Next suppose $D \neq \emptyset$. Note D is compact and $D \cap \partial U = \emptyset$, so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to 2^Y$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x)$ and let $R_\mu^\star = I \times (L+T)^{-1}(R_\mu + T)$. Notice $R_\mu \in A(\overline{U}, Y; L, T)$, $R_\mu|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ since $\mu(\partial U) =$

0, and $R_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$. Also since $\mu(D) = 1$ it is easy to see that $\left(R_\mu^\star\right)^{-1}(B) = (G^\star)^{-1}(B)$, so $d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) = d(\emptyset)$. Also note $R_\mu \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ (to see this let $Q : \overline{U} \times [0,1] \to 2^Y$ be given by $Q(x,t) = H(x, t\,\mu(x)))$, so $d\left((F^\star)^{-1}(B)\right) = d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d(\emptyset)$, and so $F^\star$ in d–L–Φ–inessential. $\square$

We have immediately the following result.

Theorem 2.4. *Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $B = \{(x, (L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U}\}$, $L : dom\, L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, d a map defined in* (2.7) *and assume* (2.8), (2.9) *and* (2.10) *hold. Suppose J and Ψ are two maps in $A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and $\Psi^\star = I \times (L+T)^{-1}(\Psi+T)$ and with $J \cong \Psi$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Then $J^\star$ is d–L–Φ–inessential if and only if $\Psi^\star$ is d–L–Φ–inessential.*

Remark 2.2. If E is a normal topological vector space then the assumption that

$$\left\{x \in \overline{U} :\ (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

is compact, can be replaced by

$$\left\{x \in \overline{U} :\ (x, (L+T)^{-1}(\Phi+T)(x)) \cap H^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

is closed, in Definition 2.7.

Next we discuss the situation when (2.3) is not assumed. To obtain an analogue of Theorem 2.1 and Theorem 2.2 we change the definition of d–Φ–essential in Definition 2.4.

Definition 2.9. Let $F \in A_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to 2^{\overline{U} \times E}$ is d–Φ–essential if for every map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Otherwise $F^\star$ is d–Φ–inessential. It is immediate that this means either $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $J \in A_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((J^\star)^{-1}(B)\right)$.

Theorem 2.5. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $B = \{(x, \Phi(x)) : x \in \overline{U}\}$, d a map defined* *in* (2.1) *and assume* (2.2) *holds. Suppose* $F \in A_{\partial U}(\overline{U}, E)$ *and assume the following condition holds:*

$$(2.11)\quad \begin{cases} \textit{if there exists a map } G \in A_{\partial U}(\overline{U}, E) \textit{ with } G \cong F \textit{ in} \\ A_{\partial U}(\overline{U}, E) \textit{ and } d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right) \\ \textit{with } G^\star = I \times G,\ F^\star = I \times F, \textit{ and if } H \textit{ is the map} \\ \textit{defined in Definition 2.3 and } \mu : \overline{U} \to [0,1] \textit{ is any} \\ \textit{continuous map with } \mu(\partial U) = 0, \textit{ then} \\ \left\{x \in \overline{U} :\ (x, \Phi(x)) \cap (x, H(x, t\,\mu(x))) \neq \emptyset \textit{ for some } t \in [0,1]\right\} \\ \textit{is closed.} \end{cases}$$

Then the following are equivalent:

(i). $F^\star = I \times F : \overline{U} \to 2^{\overline{U} \times E}$ *is* d*–*Φ*–inessential;*

(ii). $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ *or there exists a map* $G \in A_{\partial U}(\overline{U}, E)$ *with* $G^\star = I \times G$ *and* $G \cong F$ *in* $A_{\partial U}(\overline{U}, E)$ *such that* $d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right)$*.*

PROOF: (i) implies (ii) is immediate. Next we prove (ii) implies (i). If $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ then trivially (i) is true. Next suppose there exists a map $G \in A_{\partial U}(\overline{U}, E)$ with $G^\star = I \times G$ and $G \cong F$ in $A_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right)$. Let $H : \overline{U} \times [0,1] \to 2^E$ with $H(\,.\,, \eta(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $H_t(x) \cap \Phi(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = G$, $H_0 = F$ and

$$\left\{x \in \overline{U} :\ (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}$$

is compact (respectively closed); here $H^\star(x,t) = (x\,,\,H(x,t))$ and $H_t(x) = H(x,t))$. Consider

$$D = \left\{x \in \overline{U} :\ (x, \Phi(x)) \cap H^\star(x,t) \neq \emptyset \text{ for some } t \in [0,1]\right\}.$$

If $D = \emptyset$ then as in Theorem 2.1 we obtain immediately that $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ i.e. $F^\star$ is d–Φ–inessential. Next suppose $D \neq \emptyset$. Note D is compact (respectively closed). Also $D \cap \partial U = \emptyset$ and there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to 2^E$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x)$ and let $R^\star_\mu = I \times R_\mu$. Note $R_\mu \in A(\overline{U}, E)$, $R_\mu|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ since $\mu(\partial U) = 0$, and $R_\mu \in A_{\partial U}(\overline{U}, E)$. Also since $\mu(D) = 1$ we have (see Theorem 2.1) $\left(R^\star_\mu\right)^{-1}(B) = (G^\star)^{-1}(B)$, so $d\left(\left(R^\star_\mu\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. Thus $d\left((F^\star)^{-1}(B)\right) \neq$

$d\left((R_\mu^\star)^{-1}(B)\right)$. Also note (as in Theorem 2.1) $R_\mu \cong F$ in $A_{\partial U}(\overline{U},E)$ (to see this let $Q:\overline{U}\times[0,1]\to 2^E$ be given by $Q(x,t)=H(x,t\,\mu(x))$). Consequently $F^\star$ is d–Φ–inessential (take $J=R_\mu$ in Definition 2.9). □

Theorem 2.6. *Let E be a completely regular (respectively normal) topological space, U an open subset of E, $B=\{(x,\Phi(x)):\ x\in\overline{U}\}$, d a map defined in* (2.1) *and assume* (2.2) *and* (2.11) *hold. Suppose R and Ψ are two maps in $A_{\partial U}(\overline{U},E)$ with $R^\star=I\times R$ and $\Psi^\star=I\times\Psi$ and with $R\cong\Psi$ in $A_{\partial U}(\overline{U},E)$. Then $R^\star$ is d–Φ–inessential if and only if $\Psi^\star$ is d–Φ–inessential.*

PROOF: Assume $R^\star$ is d–Φ–inessential.

Then (see Theorem 2.5) either $d\left((R^\star)^{-1}(B)\right)=d(\emptyset)$ or there exists a map $Q\in A_{\partial U}(\overline{U},E)$ with $Q^\star=I\times Q$ and $Q\cong R$ in $A_{\partial U}(\overline{U},E)$ such that $d\left((R^\star)^{-1}(B)\right)\neq d\left((Q^\star)^{-1}(B)\right)$.

Suppose first that $d\left((R^\star)^{-1}(B)\right)=d(\emptyset)$. There are two cases to consider, either $d\left((\Psi^\star)^{-1}(B)\right)\neq d(\emptyset)$ or $d\left((\Psi^\star)^{-1}(B)\right)=d(\emptyset)$.

Case (1). Suppose $d\left((\Psi^\star)^{-1}(B)\right)\neq d(\emptyset)$.

Then $d\left((R^\star)^{-1}(B)\right)\neq d\left((\Psi^\star)^{-1}(B)\right)$ and we know $R\cong\Psi$ in $A_{\partial U}(\overline{U},E)$. Now Theorem 2.5 (with $F=\Psi$ and $G=R$) guarantees that $\Psi^\star$ is d–Φ–inessential.

Case (2). Suppose $d\left((\Psi^\star)^{-1}(B)\right)=d(\emptyset)$.

Then by definition $\Psi^\star$ is d–Φ–inessential.

Next suppose there exists a map $Q\in A_{\partial U}(\overline{U},E)$ with $Q^\star=I\times Q$ and $Q\cong R$ in $A_{\partial U}(\overline{U},E)$ such that $d\left((R^\star)^{-1}(B)\right)\neq d\left((Q^\star)^{-1}(B)\right)$. Note (since $\cong$ is an equivalence relation in $A_{\partial U}(\overline{U},E)$) also that $Q\cong\Psi$ in $A_{\partial U}(\overline{U},E)$. There are two cases to consider, either $d\left((Q^\star)^{-1}(B)\right)\neq d\left((\Psi^\star)^{-1}(B)\right)$ or $d\left((Q^\star)^{-1}(B)\right)=d\left((\Psi^\star)^{-1}(B)\right)$.

Case (1). Suppose $d\left((Q^\star)^{-1}(B)\right)\neq d\left((\Psi^\star)^{-1}(B)\right)$.

Then Theorem 2.5 (with $F=\Psi$ and $G=Q$) guarantees that $\Psi^\star$ is d–Φ–inessential.

Case (2). Suppose $d\left((Q^\star)^{-1}(B)\right)=d\left((\Psi^\star)^{-1}(B)\right)$.

Then $d\left((R^\star)^{-1}(B)\right)\neq d\left((\Psi^\star)^{-1}(B)\right)$ and we know $R\cong\Psi$ in $A_{\partial U}(\overline{U},E)$. Now Theorem 2.5 (with $F=\Psi$ and $G=R$) guarantees that $\Psi^\star$ is d–Φ–

inessential.

Thus in all cases $\Psi^\star$ is d–Φ–inessential.

Similarly if $\Psi^\star$ is d–Φ–inessential then $R^\star$ is d–Φ–inessential. □

Next we discuss the situation when (2.9) is not assumed. To obtain an analogue of Theorem 2.3 and Theorem 2.4 we change the definition of d–L–Φ–essential in Definition 2.8.

Definition 2.10. Let $F \in A_{\partial U}(\overline{U}, Y; L, T)$ with $F^\star = I \times (L+T)^{-1}(F+T)$. We say $F^\star : \overline{U} \to 2^{\overline{U}\times E}$ is d–L–Φ–essential if for every map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ we have that $d\left((F^\star)^{-1}(B)\right) = d\left((J^\star)^{-1}(B)\right) \neq d(\emptyset)$. Otherwise $F^\star$ is d–L–Φ–inessential. It is immediate that this means either $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $J \in A_{\partial U}(\overline{U}, Y; L, T)$ with $J^\star = I \times (L+T)^{-1}(J+T)$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((J^\star)^{-1}(B)\right)$.

Theorem 2.7. *Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $B = \left\{(x, (L+T)^{-1}(\Phi+T)(x)) : x \in \overline{U}\right\}$, $L : dom\, L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, d a map defined in (2.7) and assume (2.8) holds. Suppose $F \in A_{\partial U}(\overline{U}, Y; L, T)$ and assume the following condition holds:*

$$(2.12)\quad \begin{cases} \text{if there exists a map } G \in A_{\partial U}(\overline{U}, Y; L, T) \text{ with } G \cong F \text{ in} \\ A_{\partial U}(\overline{U}, Y; L, T) \text{ and } d\left((G^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right) \text{ with} \\ G^\star = I \times (L+T)^{-1}(G+T),\ F^\star = I \times (L+T)^{-1}(F+T), \\ \text{and if } H \text{ is the map defined in Definition 2.7 and} \\ \mu : \overline{U} \to [0,1] \text{ is any continuous map with } \mu(\partial U) = 0, \text{ then} \\ \{x \in \overline{U} :\ \emptyset \neq (x, (L+T)^{-1}(\Phi+T)(x)) \\ \quad \cap\, (x, (L+T)^{-1}(H_{t\,\mu(x)}+T)(x)) \text{ for some } t \in [0,1]\} \\ \text{is closed.} \end{cases}$$

Then the following are equivalent:

(i). $F^\star = I \times (L+T)^{-1}(F+T) : \overline{U} \to 2^{\overline{U}\times E}$ is d–L–Φ–inessential;

(ii). $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G^\star = I \times (L+T)^{-1}(G+T)$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((F^\star)^{-1}(B)\right) \neq d\left((G^\star)^{-1}(B)\right)$.

PROOF: (i) implies (ii) is immediate. Next we prove (ii) implies (i). If

$d\left((F^\star)^{-1}\,(B)\right) = d(\emptyset)$ then trivially (i) is true. Next suppose there exists a map $G \in A_{\partial U}(\overline{U}, Y; L, T)$ with $G^\star = I \times (L+T)^{-1}\,(G+T)$ and $G \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ such that $d\left((F^\star)^{-1}\,(B)\right) \neq d\left((G^\star)^{-1}\,(B)\right)$. Let $H : \overline{U} \times [0,1] \to 2^Y$ be a map with $(L+T)^{-1}\,(H(\,.\,,\eta(\,.\,)) + T(\,.\,)) \in A(\overline{U}, E)$ for any continuous function $\eta : \overline{U} \to [0,1]$ with $\eta(\partial U) = 0$, $(L+T)^{-1}\,(H_t + T)(x) \cap (L+T)^{-1}\,(\Phi + T)(x) = \emptyset$ for any $x \in \partial U$ and $t \in [0,1]$, $H_1 = G$, $H_0 = F$ (here $H_t(x) = H(x,t)$) and

$$\left\{x \in \overline{U} :\ (x, (L+T)^{-1}\,(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

is compact; here $H^\star(x,\lambda) = (x\,,\,(L+T)^{-1}\,(H+T)(x,\lambda))$.

Let

$$D = \left\{x \in \overline{U} :\ (x, (L+T)^{-1}\,(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}.$$

If $D = \emptyset$ then as in Theorem 2.3 we have $d\left((F^\star)^{-1}\,(B)\right) = d(\emptyset)$ so $F^\star$ in d–L–Φ–inessential. Next suppose $D \neq \emptyset$. Note D is compact and $D \cap \partial U = \emptyset$, so there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to 2^Y$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x)$ and let $R_\mu^\star = I \times (L+T)^{-1}\,(R_\mu + T)$. Notice $R_\mu \in A(\overline{U}, Y; L, T)$, $R_\mu|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ since $\mu(\partial U) = 0$, and $R_\mu \in A_{\partial U}(\overline{U}, Y; L, T)$. Also since $\mu(D) = 1$ we have $\left(R_\mu^\star\right)^{-1}\,(B) = (G^\star)^{-1}\,(B)$, so $d\left(\left(R_\mu^\star\right)^{-1}\,(B)\right) = d\left((G^\star)^{-1}\,(B)\right)$. Thus $d\left((F^\star)^{-1}\,(B)\right) \neq d\left(\left(R_\mu^\star\right)^{-1}\,(B)\right)$. Also note $R_\mu \cong F$ in $A_{\partial U}(\overline{U}, Y; L, T)$ (to see this let $Q : \overline{U} \times [0,1] \to 2^Y$ be given by $Q(x,t) = H(x, t\,\mu(x))$). Consequently $F^\star$ is d–L–Φ–inessential. $\square$

Theorem 2.8. *Let E be a Hausdorff topological vector space, Y a topological vector space, U an open subset of E, $B = \{(x, (L+T)^{-1}\,(\Phi + T)(x)) :\ x \in \overline{U}\}$, $L : dom\,L \subseteq E \to Y$ a linear single valued map, $T \in H_L(E, Y)$, d a map defined in* (2.7) *and assume* (2.8)*, and* (2.12) *hold. Suppose R and Ψ are two maps in $A_{\partial U}(\overline{U}, Y; L, T)$ with $R^\star = I \times (L+T)^{-1}\,(R+T)$ and $\Psi^\star = I \times (L+T)^{-1}\,(\Psi + T)$ and with $R \cong \Psi$ in $A_{\partial U}(\overline{U}, Y; L, T)$. Then $R^\star$ is d–L–Φ–inessential if and only if $\Psi^\star$ is d–L–Φ–inessential.*

Remark 2.3. If E is a normal topological vector space then the assumption that

$$\left\{x \in \overline{U} :\ (x, (L+T)^{-1}\,(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

is compact, can be replaced by

$$\left\{x \in \overline{U} :\ (x, (L+T)^{-1}\,(\Phi + T)(x)) \cap H^\star(x,t) \neq \emptyset \ \text{ for some } \ t \in [0,1]\right\}$$

is closed, in Definition 2.7.

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