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A relaxation theorem for a differential inclusion with "maxima"

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Abstract

We consider a Cauchy problem associated to a nonconvex differential inclusion with "maxima" and we prove a Filippov type existence result. This result allows to obtain a relaxation theorem for the problem considered.

1 Indroduction

Differential equations with maximum have proved to be strong tools in the modelling of many physical problems: systems with automatic regulation, problems in control theory that correspond to the maximal deviation of the regulated quantity etc.. As a consequence there was an intensive development of the theory of differential equations with "maxima" [2, 5, 6, 8-14] etc..

A classical example is the one of an electric generator ([2]). In this case the mechanism becomes active when the maximum voltage variation is reached in an interval of time. The equation describing the action of the regulator has the form

$$x'(t) = ax(t) + b \max_{s\in[t-h,t]} x(s) + f(t),$$

where a, b are constants given by the system, $x(.)$ is the voltage and $f(.)$ is a perturbation given by the change of voltage.

In this paper we study the following problem

$$x'(t) \in F(t, x(t), \max_{s\in[0,t]} x(s)) \quad a.e.\ ([0,1]), \quad x(0) = x_0 \tag{1}$$

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where $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map and $x_0 \in \mathbb{R}$. Several existing results for problem (1) obtained with fixed point approaches may be found in our previous paper [3].

The aim of this note is to obtain a relaxation theorem for the problem considered. Namely, we prove that the solution set of the problem (1) is dense in the set of the relaxed solutions; i.e. the set of solutions of the differential inclusion whose right hand side is the convex hull of the original set-valued map. In order to prove this result we show, first, that Filippov's ideas ([4]) can be suitably adapted in order to obtain the existence of solutions of problem (1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([4]) consists in proving the existence of a solution starting from a given "quasi" solution. Moreover, the result provides an estimate between the starting "quasi"solution and the solution of the differential inclusion.

The paper is organized as follows: in Section 2 we briefly recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2 Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B); a \in A\}$, where $d(x, B) = \inf\{d(x, y); y \in B\}$. Let $I := [0, 1]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Denote by $\mathcal{P}(\mathbb{R})$ the family of all nonempty subsets of $\mathbb{R}$ and by $\mathcal{B}(\mathbb{R})$ the family of all Borel subsets of $\mathbb{R}$. For any subset $A \subset \mathbb{R}$ we denote by $\mathrm{cl}A$ the closure of A and by $\overline{\mathrm{co}}(A)$ the closed convex hull of A.

As usual, we denote by $C(I, \mathbb{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbb{R}$ endowed with the norm $|x|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, \mathbb{R})$ the Banach space of all integrable functions $x(.) : I \to \mathbb{R}$ endowed with the norm $|x|_1 = \int_0^T |x(t)| \mathrm{d}t$. The Banach space of all absolutely continuous functions $x(.) : I \to \mathbb{R}$ will be denoted by $AC(I, \mathbb{R})$. We recall that for a set-valued map $U : I \to \mathcal{P}(\mathbb{R})$ the Aumann integral of U, denoted by $\int_I U(t) \mathrm{d}t$, is the set

$$\int_I U(t) \mathrm{d}t = \{\int_I u(t) \mathrm{d}t;\ u(.) \in L^1(I, \mathbb{R}),\ u(t) \in U(t)\, a.e.\, (I)\}$$

We recall two results that we are going to use in the next section. The first one is a selection result (e.g., [1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem. The proof of the second one may be found in [7].

Lemma 1. *Consider X a separable Banach space, B is the closed unit ball in X, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X, L : I \to \mathbb{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \emptyset \quad a.e.(I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

Lemma 2. *Let $U : I \to \mathcal{P}(\mathbb{R})$ be a measurable set-valued map with closed nonempty images and having at least one integrable selection. Then*

$$\mathrm{cl}(\int_0^T \overline{\mathrm{co}} U(t)\mathrm{d}t) = \mathrm{cl}(\int_0^T U(t)\mathrm{d}t).$$

Let $I(.) : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ a set-valued map with compact convex values defined by $I(t) = [a(t), b(t)]$, where $a(.), b(.) : \mathbb{R} \to \mathbb{R}$ are continuous functions with $a(t) \leq b(t)\ \forall t \in \mathbb{R}$. For $x(.) : \mathbb{R} \to \mathbb{R}$ continuous we define $(\max_I)(t) = \max_{s \in I(t)} x(s)$. Therefore, $\max_I : C(\mathbb{R}, \mathbb{R}) \to C(\mathbb{R}, \mathbb{R})$ is an operator whose properties are summarized in the next lemma proved in [12].

Lemma 3. *If $x(.), y(.) \in C(\mathbb{R}, \mathbb{R})$, then one has*

i) $|\max_{s \in I(t)} x(s) - \max_{s \in I(t)} y(s)| \leq \max_{s \in I(t)} |x(s) - y(s)|\ \forall t \in \mathbb{R}$.

ii) $\max_{t \in K} |\max_{s \in I(t)} x(s) - \max_{s \in I(t)} y(s)| \leq \max_{s \in \cup_{t \in K} I(t)} |x(s) - y(s)|$ $\forall t \in \mathbb{R}$.

3 The main results

In what follows we assume the following hypotheses.

Hypothesis. i) $F(.,.,.) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R})$ measurable.

ii) There exist $l_1(.), l_2(.) \in L^1(I, \mathbb{R}_+)$ such that, for almost all $t \in I$,

$$\mathrm{d}_H(F(t, x_1, y_1), F(t, x_2, y_2)) \leq l_1(t)|x_1 - x_2| + l_2(t)|y_1 - y_2|\ \forall x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

Theorem 1. *Assume Hypothesis satisfied and $|l_1|_1 + |l_2|_1 < 1$. Let $y(.) \in AC(I, \mathbb{R})$ be such that there exists $p(.) \in L^1(I, \mathbb{R}_+)$ verifying $d(y'(t), F(t, y(t), \max_{s \in [0,t]} y(s))) \leq p(t)$ a.e. (I).*

Then there exists $x(.)$ a solution of problem (1) satisfying for all $t \in I$

$$|x - y|_C \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}(|x_0 - y(0)| + |p|_1). \tag{2}$$

Proof. We set $x_0(.) = y(.)$, $f_0(.) = y'(.)$.

The set-valued map $t \to F(t, y(t), \max_{s\in[0,t]} y(s))$ is measurable with closed values and

$$F(t, y(t), \max_{s\in[0,t]} y(s)) \cap \{y'(t) + p(t)[-1, 1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 1 that there exists a measurable function $f_1(.)$ such that $f_1(t) \in F(t, x_0(t), \max_{s\in[0,t]} x_0(s))$ a.e. (I) and, for almost all $t \in I$, $|f_1(t) - y'(t)| \leq p(t)$. Define $x_1(t) = x_0 + \int_0^t f_1(s)ds$ and one has

$$|x_1(t) - y(t)| \leq |x_0 - y(0)| + \int_0^t p(s)ds \leq |x_0 - y(0)| + |p|_1.$$

Thus $|x_1 - y|_C \leq |x_0 - y(0)| + |p|_1$.

The set-valued map $t \to F(t, x_1(t), \max_{s\in[0,t]} x_1(s))$ is measurable. Moreover, the map $t \to l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s\in[0,t]} x_1(s) - \max_{s\in[0,t]} x_0(s)|$ is measurable. By the lipschitzianity of $F(t, ., .)$ we have that for almost all $t \in I$

$$d(f_1(t), F(t, x_1(t), \max_{s\in[0,t]} x_1(s))) \leq d_H(F(t, x_0(t), \max_{s\in[0,t]} x_0(s)),$$

$$F(t, x_1(t), \max_{s\in[0,t]} x_1(s))) \leq l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s\in[0,t]} x_0(s) - \max_{s\in[0,t]} x_1(s)|.$$

Therefore,

$$F(t, x_1(t), \max_{s\in[0,t]} x_1(s))) \cap \{f_1(t) + (l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s\in[0,t]} x_1(s) - \max_{s\in[0,t]} x_0(s)|)[-1, 1]\} \neq \emptyset.$$

From Lemma 1 we deduce the existence of a measurable function $f_2(.)$ such that $f_2(t) \in F(t, x_1(t), \max_{s\in[0,t]} x_1(s))$ a.e. (I) and for almost all $t \in I$

$$|f_1(t) - f_2(t)| \leq d(f_1(t), F(t, x_1(t), \max_{s\in[0,t]} x_1(s))) \leq d_H(F(t, x_0(t), \max_{s\in[0,t]} x_0(s)),$$

$$F(t, x_1(t), \max_{s\in[0,t]} x_1(s))) \leq l_1(t)|x_1(t) - x_0(t)| + l_2(t)|\max_{s\in[0,t]} x_0(s) - \max_{s\in[0,t]} x_1(s)|.$$

Define $x_2(t) = x_0 + \int_0^t f_2(s)ds$ and one has

$$|x_1(t) - x_2(t)| \leq \int_0^t |f_1(s) - f_2(s)|ds \leq \int_0^t [l_1(s)|x_0(s) - x_1(s)| +$$

$$l_2(s)|\max_{\sigma\in[0,s]} x_0(\sigma) - \max_{\sigma\in[0,s]} x_1(\sigma)|]ds \leq (|l_1|_1 + |l_2|_1)|x_1 - x_0|_C$$

$$\leq (|l_1|_1 + |l_2|_1)(|x_0 - y(0)| + |p|_1).$$

Assume that for some $n \geq 1$ we have constructed $(x_i(.))_{i=1}^n$ with x_n satisfying

$$|x_n - x_{n-1}|_C \leq (|l_1|_1 + |l_2|_1)^{n-1}(|x_0 - y(0)| + |p|_1).$$

The set-valued map $t \to F(t, x_n(t), \max_{s\in[0,t]} x_n(s))$ is measurable. At the same time, the map $t \to l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s\in[0,t]} x_n(s) - \max_{s\in[0,t]} x_{n-1}(s)|$ is measurable. As before, by the lipschitzianity of $F(t, ., .)$ we have that for almost all $t \in I$

$$F(t, x_n(t), \max\nolimits_{s\in[0,t]} x_n(s))) \cap \{f_n(t) + (l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max\nolimits_{s\in[0,t]} x_n(s) - \max\nolimits_{s\in[0,t]} x_{n-1}(s)|)[-1, 1]\} \neq \emptyset.$$

Using again Lemma 1 we deduce the existence of a measurable function $f_{n+1}(.)$ such that $f_{n+1}(t) \in F(t, x_n(t), \max_{s\in[0,t]} x_n(s))$ a.e. (I) and for almost all $t \in I$

$$|f_{n+1}(t) - f_n(t)| \leq d(f_{n+1}(t), F(t, x_{n-1}(t), \max_{s\in[0,t]} x_{n-1}(s))) \leq$$

$$d_H(F(t, x_n(t), \max_{s\in[0,t]} x_n(s)), F(t, x_{n-1}(t), \max_{s\in[0,t]} x_{n-1}(s))) \leq$$

$$l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s\in[0,t]} x_n(s) - \max_{s\in[0,t]} x_{n-1}(s)|.$$

Define

$$x_{n+1}(t) = x_0 + \int_0^t f_{n+1}(s)ds. \tag{3}$$

We have

$$|x_{n+1}(t) - x_n(t)| \leq \int_0^t |f_{n+1}(s) - f_n(s)|ds \leq$$

$$\int_0^t [l_1(s)|x_n(s) - x_{n-1}(s)| + l_2(s)|\max_{\sigma\in[0,s]} x_n(\sigma) - \max_{\sigma\in[0,s]} x_{n-1}(\sigma)|]ds$$

$$\leq (|l_1|_1 + |l_2|_1)|x_n - x_{n-1}|_C \leq (|l_1|_1 + |l_2|_1)^n(|x_0 - y(0)| + |p|_1).$$

Therefore $(x_n(.))_{n\geq 0}$ is a Cauchy sequence in the Banach space $C(I, \mathbb{R})$, so it converges to $x(.) \in C(I, \mathbb{R})$. Since, for almost all $t \in I$, we have

$$|f_{n+1}(t) - f_n(t)| \leq l_1(t)|x_n(t) - x_{n-1}(t)| + l_2(t)|\max_{s\in[0,t]} x_n(t) - \max_{s\in[0,t]} x_{n-1}(t)|$$

$$\leq [l_1(t) + l_2(t)]|x_n - x_{n-1}|_C,$$

$\{f_n(.)\}$ is a Cauchy sequence in the Banach space $L^1(I, \mathbb{R})$ and thus it converges to $f(.) \in L^1(I, \mathbb{R})$.

We note that one may write

$$|\int_0^t f_n(s)ds - \int_0^t f(s)ds| \leq \int_0^t |f_n(s)-f(s)|ds \leq \int_0^t [l_1(s)+l_2(s)]|x_{n+1}-x|_C ds$$
$$\leq (|l_1|_1 + |l_2|_1).|x_{n+1} - x|_C.$$

Therefore, one may pass to the limit in (3) and we get $x(t) = x_0 + \int_0^t f(s)ds$. Moreover, since the values of $F(.,.,.)$ are closed and $f_{n+1}(t) \in F(t, x_n(t), \max_{s\in[0,t]} x_n(t))$ passing to the limit we obtain $f(t) \in F(t, x(t), \max_{s\in[0,t]} x(t))$ a.e. (I).

It remains to prove the estimate (2). One has

$$|x_n - x_0|_C \leq |x_n - x_{n-1}|_C + ... + |x_2 - x_1|_C + |x_1 - x_0|_C \leq$$
$$(|l_1|_1 + |l_2|_1)^{n-1}(|x_0 - y(0)| + |p|_1) + ... + (|l_1|_1 + |l_2|_1)(|x_0 - y(0)| + |p|_1)+$$
$$(|x_0 - y(0)| + |p|_1) \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}(|x_0 - y(0)| + |p|_1).$$

Passage to the limit in the last inequality completes the proof. □

Remark 1. A similar result to the one in Theorem 1 may be found in [3], namely Theorem 3.1. The approach in [3], apart from the requirement that the values of $F(.,.)$ are compact, does not provides a priori bounds for solutions as in (3.1).

As we already pointed out, Theorem 1 allows to obtain a relaxation theorem for problem (1). In what follows, we are concerned also with the convexified (relaxed) problem

$$x'(t) \in \overline{\text{co}}F(t, x(t), \max_{s\in[0,t]} x(s)), \quad x(0) = x_0. \tag{4}$$

Note that if $F(.,.,.)$ satisfies Hypothesis, then so does the set-valued map $(t, x, y) \to \overline{\text{co}}F(t, x, y)$ (e.g., [1]).

Theorem 2. *We assume that Hypothesis is satisfied and* $|l_1|_1 + |l_2|_1 < 1$. *Let* $\overline{x}(.) : I \to \mathbb{R}$ *be a solution to the relaxed inclusion (4) such that the set-valued map* $t \to F(t, \overline{x}(t), \max_{s\in[0,t]} \overline{x}(s))$ *has at least one integrable selection.*

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of problem (1) such that*

$$|x - \overline{x}|_C < \varepsilon.$$

Proof. Since $\overline{x}(.)$ is a solution of the relaxed inclusion (4), there exists $\overline{f}(.) \in L^1(I, \mathbb{R})$, $\overline{f}(t) \in \overline{\text{co}}F(t, \overline{x}(t), \max_{s\in[0,t]} \overline{x}(s))$ a.e. (I) such that $\overline{x}(t) = x_0 + \int_0^t \overline{f}(s)ds$.

From Lemma 2, for $\delta > 0$, there exists $\tilde{f}(t) \in F(t, \overline{x}(t), \max_{s\in[0,t]} \overline{x}(s))$ a.e. (I) such that

$$\sup_{t\in I} |\int_0^t (\tilde{f}(s) - \overline{f}(s))ds| \leq \delta.$$

Define $\tilde{x}(t) = x_0 + \int_0^t \tilde{f}(s)ds$. Therefore, $|\tilde{x} - \overline{x}|_C \leq \delta$.

We apply Theorem 1 for the "quasi" solution $\tilde{x}(.)$ of (1). One has

$$p(t) = d(\tilde{f}(t), F(t, \tilde{x}(t), \max_{s\in[0,t]} \tilde{x}(s))) \leq d_H(F(t, \overline{x}(t), \max_{s\in[0,t]} \overline{x}(s)),$$

$$F(t, \tilde{x}(t), \max_{s\in[0,t]} \tilde{x}(s))) \leq l_1(t)|\overline{x}(t) - \tilde{x}(t)| + l_2(t)| \max_{s\in[0,t]} \overline{x}(s) - \max_{s\in[0,t]} \tilde{x}(s)|$$

$$\leq l_1(t)|\tilde{x} - \overline{x}|_C + l_2(t)|\tilde{x} - \overline{x}|_C \leq (l_1(t) + l_2(t))\delta,$$

which shows that $p(.) \in L^1(I, \mathbb{R})$.

From Theorem 1 there exists $x(.)$ a solution of (1) such that

$$|x - \tilde{x}|_C \leq \frac{1}{1 - (|l_1|_1 + |l_2|_1)}|p|_1 \leq \frac{|l_1|_1 + |l_2|_1}{1 - (|l_1|_1 + |l_2|_1)}\delta.$$

It remains to take $\delta = [1 - (|l_1|_1 + |l_2|_1)]\varepsilon$ and to deduce that $|x - \overline{x}|_C \leq |x - \tilde{x}|_C + |\tilde{x} - \overline{x}|_C \leq \varepsilon$. □

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