



Ostrowski Via a Two Functions Pompeiu's Inequality

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Abstract

In this paper, some generalizations of Pompeiu's inequality for two complex-valued absolutely continuous functions are provided. They are applied to obtain some new Ostrowski type results. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

1 Introduction

In 1946, Pompeiu [6] derived a variant of *Lagrange's mean value theorem*, now known as *Pompeiu's mean value theorem* (see also [8, p. 83]).

Theorem 1 (Pompeiu, 1946 [6]). *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ between x_1 and x_2 such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \quad (1)$$

The following inequality is useful to derive some Ostrowski type inequalities.

Key Words: Ostrowski's inequality, Pompeiu's mean value theorem, Integral inequalities, Schwarz inequality

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Corollary 1 (Pompeiu's Inequality). *With the assumptions of Theorem 1 and if $\|f - \ell f'\|_\infty = \sup_{t \in (a,b)} |f(t) - \ell f'(t)| < \infty$ where $\ell(t) = t$, $t \in [a, b]$, then*

$$|tf(x) - xf(t)| \leq \|f - \ell f'\|_\infty |x - t| \quad (2)$$

for any $t, x \in [a, b]$.

The inequality (2) was stated by the author in [3].

In 1938, A. Ostrowski [4] proved the following result in the estimating the integral mean:

Theorem 2 (Ostrowski, 1938 [4]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $|f'(t)| \leq M < \infty$ for all $t \in (a, b)$. Then for any $x \in [a, b]$, we have the inequality*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] M (b-a). \quad (3)$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller quantity.

In order to provide another approximation of the integral mean, by making use of the Pompeiu's mean value theorem, the author proved the following result:

Theorem 3 (Dragomir, 2005 [3]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have the inequality*

$$\begin{aligned} & \left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f - \ell f'\|_\infty, \end{aligned} \quad (4)$$

where $\ell(t) = t$, $t \in [a, b]$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

In [7], E. C. Popa using a mean value theorem obtained a generalization of (4) as follows:

Theorem 4 (Popa, 2007 [7]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . Assume that $\alpha \notin [a, b]$. Then for any $x \in [a, b]$, we have the inequality*

$$\begin{aligned} & \left| \left(\frac{a+b}{2} - \alpha \right) f(x) + \frac{\alpha-x}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f - \ell_\alpha f'\|_\infty, \end{aligned} \quad (5)$$

where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$.

In [5], J. Pečarić and S. Ungar have proved a general estimate with the p -norm, $1 \leq p \leq \infty$ which for $p = \infty$ gives Dragomir's result.

Theorem 5 (Pečarić & Ungar, 2006 [5]). *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq PU(x, p) \|f - \ell f'\|_p, \quad (6)$$

for $x \in [a, b]$, where

$$\begin{aligned} PU(x, p) & : = (b-a)^{\frac{1}{p}-1} \left[\left(\frac{a^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - a^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right. \\ & \quad \left. + \left(\frac{b^{2-q} - x^{2-q}}{(1-2q)(2-q)} + \frac{x^{2-q} - b^{1+q}x^{1-2q}}{(1-2q)(1+q)} \right)^{1/q} \right]. \end{aligned}$$

In the cases $(p, q) = (1, \infty), (\infty, 1)$ and $(2, 2)$ the quantity $PU(x, p)$ has to be taken as the limit as $p \rightarrow 1, \infty$ and 2 , respectively.

For other inequalities in terms of the p -norm of the quantity $f - \ell_\alpha f'$, where $\ell_\alpha(t) = t - \alpha$, $t \in [a, b]$ and $\alpha \notin [a, b]$ see [1] and [2].

In this paper, some new Pompeiu's type inequalities for two complex-valued absolutely continuous functions are provided. They are applied to obtain some Ostrowski type inequalities. Reverses for the integral Cauchy-Bunyakovsky-Schwarz inequality are provided as well.

2 A General Pompeiu's Inequality

We start with the following generalization of (2).

Theorem 6. Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $g(t) \neq 0$ for all $t \in [a, b]$. Then for any $t, x \in [a, b]$ we have

$$\left| \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right| \leq \begin{cases} \|f'g - fg'\|_\infty \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_\infty[a, b], \\ \|f'g - fg'\|_p \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'g - fg'\|_1 \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|g(s)|^2} \right\} & \end{cases} \quad (7)$$

or, equivalently

$$|g(t)f(x) - f(t)g(x)| \leq \begin{cases} \|f'g - fg'\|_\infty |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'g - fg' \in L_\infty[a, b], \\ \|f'g - fg'\|_p |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'g - fg'\|_1 |g(t)g(x)| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|g(s)|^2} \right\}. & \end{cases} \quad (8)$$

Proof. If f and g are absolutely continuous and $g(t) \neq 0$ for all $t \in [a, b]$, then f/g is absolutely continuous on the interval $[a, b]$ and

$$\int_t^x \left(\frac{f(s)}{g(s)} \right)' ds = \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)}$$

for any $t, x \in [a, b]$ with $x \neq t$.

Since

$$\int_t^x \left(\frac{f(s)}{g(s)} \right)' ds = \int_t^x \frac{f'(s)g(s) - f(s)g'(s)}{g^2(s)} ds,$$

then we get the following identity

$$\frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} = \int_t^x \frac{f'(s)g(s) - f(s)g'(s)}{g^2(s)} ds \quad (9)$$

for any $t, x \in [a, b]$.

Taking the modulus in (9) we have

$$\begin{aligned} \left| \frac{f(x)}{g(x)} - \frac{f(t)}{g(t)} \right| &= \left| \int_t^x \frac{f'(s)g(s) - f(s)g'(s)}{g^2(s)} ds \right| \\ &\leq \left| \int_t^x \frac{|f'(s)g(s) - f(s)g'(s)|}{|g(s)|^2} ds \right| := I \end{aligned} \tag{10}$$

and utilizing Hölder's integral inequality we deduce

$$\begin{aligned} I &\leq \begin{cases} \sup_{s \in [t,x]([x,t])} |f'(s)g(s) - f(s)g'(s)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\ \left| \int_t^x |f'(s)g(s) - f(s)g'(s)|^p ds \right|^{1/p} \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ \left| \int_t^x |f'(s)g(s) - f(s)g'(s)| ds \right| \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\}, \end{cases} \\ &\leq \begin{cases} \|f'g - fg'\|_\infty \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|, \\ \|f'g - fg'\|_p \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \begin{matrix} p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \\ \|f'g - fg'\|_1 \sup_{s \in [t,x]([x,t])} \left\{ \frac{1}{|g(s)|^2} \right\} \end{cases} \end{aligned}$$

and the inequality (7) is proved. □

The following particular case extends Pompeiu's inequality to other p -norms than $p = \infty$ obtained in (2).

Corollary 2. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. Then for any $t, x \in [a, b]$ we have*

$$\begin{aligned}
& \left| \frac{f(x)}{x} - \frac{f(t)}{t} \right| \\
& \leq \begin{cases} \|f - \ell f'\|_\infty \left| \frac{1}{t} - \frac{1}{x} \right| & \text{if } f - \ell f' \in L_\infty[a, b], \\ \frac{1}{(2q-1)^{1/q}} \|f - \ell f'\|_p \left| \frac{1}{t^{2q-1}} - \frac{1}{x^{2q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f - \ell f'\|_1 \frac{1}{\min\{t^2, x^2\}} & \end{cases} \quad (11)
\end{aligned}$$

or, equivalently

$$\begin{aligned}
& |tf(x) - xf(t)| \\
& \leq \begin{cases} \|f - \ell f'\|_\infty |x - t| & \text{if } f - \ell f' \in L_\infty[a, b], \\ \frac{1}{(2q-1)^{1/q}} \|f - \ell f'\|_p \left| \frac{x^q}{t^{q-1}} - \frac{t^q}{x^{q-1}} \right|^{1/q} & \text{if } f - \ell f' \in L_p[a, b] \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \|f - \ell f'\|_1 \frac{\max\{t, x\}}{\min\{t, x\}}, & \end{cases} \quad (12)
\end{aligned}$$

where $\ell(t) = t, t \in [a, b]$.

The proof follows by (7) for $g(t) = \ell(t) = t, t \in [a, b]$.

The general case for power functions is as follows.

Corollary 3. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ with $b > a > 0$. If $r \in \mathbb{R}, r \neq 0$, then for any $t, x \in [a, b]$ we have*

$$\left| \frac{f(x)}{x^r} - \frac{f(t)}{t^r} \right| \leq \begin{cases} \frac{1}{|r|} \|f'\ell - rf\|_\infty \left| \frac{1}{x^r} - \frac{1}{t^r} \right|, & \text{if } f'\ell - rf \in L_\infty[a, b], \\ \|f'\ell - rf\|_p \\ \times \begin{cases} \frac{1}{|1-q(r+1)|^{1/q}} \left| \frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}} \right|^{1/q}, & \text{for } r \neq -\frac{1}{p} \\ |\ln x - \ln t|^{1/q}, & \text{for } r = -\frac{1}{p} \end{cases} \\ \text{if } f'\ell - rf \in L_p[a, b], \\ \|f'\ell - rf\|_1 \frac{1}{\min\{x^{r+1}, t^{r+1}\}}, \end{cases} \quad (13)$$

or, equivalently

$$|t^r f(x) - x^r f(t)| \leq \begin{cases} \frac{1}{|r|} \|f'\ell - rf\|_\infty |t^r - x^r|, & \text{if } f'\ell - rf \in L_\infty[a, b], \\ \|f'\ell - rf\|_p \\ \times \begin{cases} \frac{t^r x^r}{|1-q(r+1)|^{1/q}} \left| \frac{1}{x^{1-q(r+1)}} - \frac{1}{t^{1-q(r+1)}} \right|^{1/q}, & \text{for } r \neq -\frac{1}{p} \\ t^r x^r |\ln x - \ln t|^{1/q}, & \text{for } r = -\frac{1}{p} \end{cases} \\ \text{if } f'\ell - rf \in L_p[a, b], \\ \|f'\ell - rf\|_1 \frac{t^r x^r}{\min\{x^{r+1}, t^{r+1}\}}, \end{cases} \quad (14)$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

The proof follows by (7) for $g(t) = t^r$, $t \in [a, b]$. The details for calculations are omitted.

We have the following result for exponential.

Corollary 4. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the*

interval $[a, b]$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then for any $t, x \in [a, b]$ we have

$$\left| \frac{f(x)}{\exp(i\alpha x)} - \frac{f(t)}{\exp(i\alpha t)} \right| \leq \begin{cases} \|f' - i\alpha f\|_{\infty} |x - t| & \text{if } f' - i\alpha f \in L_{\infty}[a, b], \\ \|f' - i\alpha f\|_p |x - t|^{1/q} & \text{if } f' - i\alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - i\alpha f\|_1 & \end{cases} \quad (15)$$

or, equivalently

$$|\exp(i\alpha t) f(x) - f(t) \exp(i\alpha x)| \leq \begin{cases} \|f' - i\alpha f\|_{\infty} |x - t| & \text{if } f' - i\alpha f \in L_{\infty}[a, b], \\ \|f' - i\alpha f\|_p |x - t|^{1/q} & \text{if } f' - i\alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - i\alpha f\|_1. & \end{cases} \quad (16)$$

3 An Inequality Generalizing Ostrowski's

The following result holds:

Theorem 7. *Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$. If $0 < m \leq |g(t)| \leq M < \infty$ for any $t \in [a, b]$, then*

$$\begin{aligned}
& \left| f(x) \int_a^b g(t) dt - g(x) \int_a^b f(t) dt \right| \\
& \leq \left(\frac{M}{m} \right)^2 \begin{cases} \|f'g - fg'\|_\infty (b-a)^2 \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] & \text{if } f'g - fg' \in L_\infty[a, b], \\ \|f'g - fg'\|_p \left[\frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q} \right] & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'g - fg'\|_1 (b-a) & \end{cases}
\end{aligned} \tag{17}$$

for any $x \in [a, b]$.

Proof. Utilizing (8) we have

$$\begin{aligned}
& \left| f(x) \int_a^b g(t) dt - g(x) \int_a^b f(t) dt \right| \\
& \leq \int_a^b |g(t) f(x) - f(t) g(x)| dt \\
& \leq \begin{cases} \|f'g - fg'\|_\infty |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \|f'g - fg'\|_p |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \|f'g - fg'\|_1 |g(x)| \int_a^b \left(|g(t)| \sup_{s \in [t, x] \cap [x, t]} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases}
\end{aligned} \tag{18}$$

for any $x \in [a, b]$, which is of interest in itself.

Since $0 < m \leq |g(t)| \leq M < \infty$ for any $t \in [a, b]$, then

$$\begin{aligned} |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt &\leq \left(\frac{M}{m} \right)^2 \int_a^b |x-t| dt \\ &= \left(\frac{M}{m} \right)^2 \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right], \end{aligned}$$

$$\begin{aligned} |g(x)| \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt \\ \leq \left(\frac{M}{m} \right)^2 \int_a^b |x-t|^{1/q} dt = \left(\frac{M}{m} \right)^2 \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q} \end{aligned}$$

and

$$|g(x)| \int_a^b \left(|g(t)| \sup_{s \in [t, x] \cup [x, t]} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \leq \left(\frac{M}{m} \right)^2 \int_a^b dt = \left(\frac{M}{m} \right)^2 (b-a)$$

for any $x \in [a, b]$ and by (18) we get the desired result (17). \square

Remark 1. If we take $g(t) = 1$, $t \in [a, b]$ in the first inequality (17) we recapture Ostrowski's inequality.

Corollary 5. *With the assumptions in Theorem 7 we have the midpoint inequalities*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2}\right) \int_a^b g(t) dt - g\left(\frac{a+b}{2}\right) \int_a^b f(t) dt \right| \\ &\leq \left(\frac{M}{m} \right)^2 \begin{cases} \frac{1}{4} (b-a)^2 \|f'g - fg'\|_\infty & \text{if } f'g - fg' \in L_\infty[a, b], \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f'g - fg'\|_p & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \end{aligned} \tag{19}$$

The following result also holds:

Theorem 8. Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$, $g(x) \neq 0$ for $x \in [a, b]$ and $g^{-2} \in L_\infty [a, b]$. Then

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t) dt - \int_a^b f(t) dt \right| \leq \|g^{-2}\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| |x-t| dt, & \text{if } f'g - fg' \in L_\infty [a, b], \\ \|f'g - fg'\|_p \int_a^b |g(t)| |x-t|^{1/q} dt & \text{if } f'g - fg' \in L_p [a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'g - fg'\|_1 \int_a^b |g(t)| dt \end{cases} \tag{20}$$

for any $x \in [a, b]$.

Proof. Utilizing (8) we have

$$\left| \frac{f(x)}{g(x)} \int_a^b g(t) dt - \int_a^b f(t) dt \right| \leq \begin{cases} \|f'g - fg'\|_\infty \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \right) dt, \\ \|f'g - fg'\|_p \int_a^b \left(|g(t)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \right) dt, \\ \|f'g - fg'\|_1 \int_a^b \left(|g(t)| \sup_{s \in [t,x] \setminus \{x,t\}} \left\{ \frac{1}{|g(s)|^2} \right\} \right) dt \end{cases} \tag{21}$$

for any $x \in [a, b]$.

Since

$$\left| \int_t^x \frac{1}{|g(s)|^2} ds \right| \leq \|g^{-2}\|_\infty |x-t|,$$

$$\left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} \leq \|g^{-2}\|_\infty |x-t|^{1/q}$$

and

$$\sup_{s \in [t,x] \setminus \{x,t\}} \left\{ \frac{1}{|g(s)|^2} \right\} \leq \|g^{-2}\|_\infty$$

for any $x, t \in [a, b]$, then on making use of (21) we get the desired result (20). \square

We have the midpoint inequalities:

Corollary 6. *With the assumptions of Theorem 8 we have*

$$\left| \frac{f\left(\frac{a+b}{2}\right)}{g\left(\frac{a+b}{2}\right)} \int_a^b g(t) dt - \int_a^b f(t) dt \right| \leq \|g^{-2}\|_\infty \times \begin{cases} \|f'g - fg'\|_\infty \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right| dt, & \text{if } f'g - fg' \in L_\infty[a, b], \\ \|f'g - fg'\|_p \int_a^b |g(t)| \left| \frac{a+b}{2} - t \right|^{1/q} dt & \text{if } f'g - fg' \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \tag{22}$$

We have the following exponential version of Ostrowski's inequality as well:

Theorem 9. *Let $f : [a, b] \rightarrow \mathbb{C}$ be an absolutely continuous function on the interval $[a, b]$ and $\alpha \in \mathbb{R}, \alpha \neq 0$. Then for any $x \in [a, b]$ we have*

$$\left| \frac{\exp(i\alpha(b-x)) - \exp(-i\alpha(x-a))}{i\alpha} f(x) - \int_a^b f(t) dt \right| \leq \begin{cases} \|f' - i\alpha f\|_\infty (b-a)^2 \left[\frac{1}{4} + \left(\frac{t - \frac{a+b}{2}}{b-a} \right)^2 \right], & \text{if } f' - i\alpha f \in L_\infty[a, b], \\ \|f' - i\alpha f\|_p \frac{(b-x)^{1+1/q} + (x-a)^{1+1/q}}{1+1/q}, & \text{if } f' - i\alpha f \in L_p[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f' - i\alpha f\|_1. \end{cases} \tag{23}$$

Proof. If we write the inequality (18) for $g(t) = \exp(i\alpha t)$, $t \in [a, b]$, then we get

$$\left| f(x) \int_a^b \exp(i\alpha t) dt - \exp(i\alpha x) \int_a^b f(t) dt \right| \leq \begin{cases} \|f' - i\alpha f\|_\infty \int_a^b |x - t| dt, & \text{if } f' - i\alpha f \in L_\infty[a, b] \\ \|f' - i\alpha f\|_p |g(x)| \int_a^b |x - t|^{1/q} dt, & \begin{matrix} \text{if } f' - i\alpha f \\ \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1 \end{matrix} \\ \|f' - i\alpha f\|_1, & \end{cases}$$

which, after simple calculation, is equivalent with (23).

The details are omitted. □

Corollary 7. *With the assumptions of Theorem 9 we have the midpoint inequalities*

$$\left| \frac{\exp\left(i\alpha\left(\frac{b-a}{2}\right)\right) - \exp\left(-i\alpha\left(\frac{b-a}{2}\right)\right)}{i\alpha} f\left(\frac{a+b}{2}\right) - \int_a^b f(t) dt \right| \leq \begin{cases} \frac{1}{4} \|f' - i\alpha f\|_\infty (b-a)^2, & \text{if } f' - i\alpha f \in L_\infty[a, b], \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f' - i\alpha f\|_p, & \begin{matrix} \text{if } f' - i\alpha f \in L_p[a, b] \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \end{matrix} \end{cases} \tag{24}$$

or, equivalently

$$\begin{aligned}
& \left| \frac{2 \sin \left(\alpha \left(\frac{b-a}{2} \right) \right)}{\alpha} f \left(\frac{a+b}{2} \right) - \int_a^b f(t) dt \right| \\
& \leq \begin{cases} \frac{1}{4} \|f' - i\alpha f\|_{\infty} (b-a)^2, & \text{if } f' - i\alpha f \in L_{\infty}[a, b], \\ \frac{1}{2^{1/q}(1+1/q)} (b-a)^{1+1/q} \|f' - i\alpha f\|_p, & \text{if } f' - i\alpha f \in L_p[a, b] \\ & p > 1, \frac{1}{p} + \frac{1}{q} = 1. \end{cases} \quad (25)
\end{aligned}$$

4 An Application for CBS-Inequality

The following inequality is well known in the literature as the Cauchy-Bunyakovsky-Schwarz inequality, or the CBS-inequality, for short:

$$\left| \int_a^b f(t) g(t) dt \right|^2 \leq \int_a^b |f(t)|^2 dt \int_a^b |g(t)|^2 dt, \quad (26)$$

provided that $f, g \in L_2[a, b]$.

We have the following result concerning some reverses of the CBS-inequality:

Theorem 10. *Let $f, g : [a, b] \rightarrow \mathbb{C}$ be absolutely continuous functions on the interval $[a, b]$ with $g(t) \neq 0$ for all $t \in [a, b]$. Then*

$$\begin{aligned}
& 0 \leq \int_a^b |g(t)|^2 dt \int_a^b |f(t)|^2 dt - \left| \int_a^b f(t) g(t) dt \right|^2 \\
& \leq \frac{1}{2} \times \begin{cases} \|f'\bar{g} - f\bar{g}'\|_{\infty}^2 \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^2} dt \right)^2, & \text{if } f'\bar{g} - f\bar{g}' \in L_{\infty}[a, b], \\ & \frac{1}{|g|^2} \in L[a, b] \\ \|f'\bar{g} - f\bar{g}'\|_p^2 \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}, & \text{if } f'\bar{g} - f\bar{g}' \in L_p[a, b], \\ & \frac{1}{|g|^{2q}} \in L[a, b] \\ & p > 1, \\ & \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'\bar{g} - f\bar{g}'\|_1^2 \left(\int_a^b |g(t)|^2 dt \right)^2 \operatorname{ess\,sup}_{t \in [a, b]} \left\{ \frac{1}{|g(t)|^4} \right\}, & \text{if } \frac{1}{|g|} \in L_{\infty}[a, b]. \end{cases} \quad (27)
\end{aligned}$$

Proof. Utilising the inequality (8) we have

$$\left| \overline{g(t)f(x)} - f(t)\overline{g(x)} \right| \leq \begin{cases} \|f'\overline{g} - f\overline{g}'\|_{\infty} |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^2} ds \right| & \text{if } f'\overline{g} - f\overline{g}' \\ \in L_{\infty}[a, b], \\ \|f'\overline{g} - f\overline{g}'\|_p |g(t)g(x)| \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{1/q} & \text{if } f'\overline{g} - f\overline{g}' \\ \in L_p[a, b] \\ p > 1, \\ \frac{1}{p} + \frac{1}{q} = 1, \\ \|f'\overline{g} - f\overline{g}'\|_1 |g(t)g(x)| \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|g(s)|^2} \right\}. \end{cases} \quad (28)$$

for any $t, x \in [a, b]$.

Taking the square in (28) and integrating over $(t, x) \in [a, b]^2$ we have

$$\int_a^b \int_a^b \left| \overline{g(t)f(x)} - f(t)\overline{g(x)} \right|^2 dt dx \leq \begin{cases} \|f'\overline{g} - f\overline{g}'\|_{\infty}^2 \int_a^b \int_a^b |g(t)g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|^2 dt dx, \\ \|f'\overline{g} - f\overline{g}'\|_p^2 \int_a^b \int_a^b |g(t)g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} dt dx, \\ \|f'\overline{g} - f\overline{g}'\|_1^2 \int_a^b \int_a^b |g(t)g(x)|^2 \sup_{s \in [t, x] \setminus \{x, t\}} \left\{ \frac{1}{|g(s)|^4} \right\} dt dx. \end{cases} \quad (29)$$

Observe that

$$\begin{aligned}
& \int_a^b \int_a^b \left| \overline{g(t)} f(x) - f(t) \overline{g(x)} \right|^2 dt dx \\
&= \int_a^b \int_a^b \left(|g(t)|^2 |f(x)|^2 - 2 \operatorname{Re} \left[\overline{g(t)} f(x) \overline{f(t) g(x)} \right] + |g(x)|^2 |f(t)|^2 \right) dt dx \\
&= \int_a^b |g(t)|^2 dt \int_a^b |f(x)|^2 dx - 2 \operatorname{Re} \left[\int_a^b \overline{f(t) g(t)} dt \int_a^b f(x) g(x) dx \right] \\
&+ \int_a^b |g(x)|^2 dx \int_a^b |f(t)|^2 dt \\
&= 2 \left[\int_a^b |g(t)|^2 dt \int_a^b |f(t)|^2 dt - \left| \int_a^b f(t) g(t) dt \right|^2 \right],
\end{aligned}$$

$$\begin{aligned}
& \int_a^b \int_a^b \left[|g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^2} ds \right|^2 \right] dt dx \\
&\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^2} dt \right)^2,
\end{aligned}$$

$$\begin{aligned}
& \int_a^b \int_a^b \left[|g(t) g(x)|^2 \left| \int_t^x \frac{1}{|g(s)|^{2q}} ds \right|^{2/q} \right] dt dx \\
&\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \left(\int_a^b \frac{1}{|g(t)|^{2q}} dt \right)^{2/q}
\end{aligned}$$

and

$$\begin{aligned}
& \int_a^b \int_a^b \left[|g(t) g(x)|^2 \sup_{s \in [t, x] \cup ([x, t])} \left\{ \frac{1}{|g(s)|^4} \right\} \right] dt dx \\
&\leq \left(\int_a^b |g(t)|^2 dt \right)^2 \operatorname{ess\,sup}_{t \in [a, b]} \left\{ \frac{1}{|g(t)|^4} \right\},
\end{aligned}$$

then by (29) we get the desired result (27). \square

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