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From lattices to H_v–matrices

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Abstract

In this paper we study the concept of sets of elements, related to results of an associative binary operation. We discuss this issue in the context of matrices and lattices. First of all, we define hyperoperations similar to those used when constructing hyperstructures from quasi-ordered semigroups. This then enables us to show that if entries of matrices are elements of lattices, these considerations provide a natural link between matrices, some basic concepts of the hyperstructure theory including H_v–rings and H_v–matrices and also one recent construction of hyperstructures.

1 Introduction

A great variety of concepts can be described by grouping together such elements of a given set that are related to the result of a binary operation defined on it. In the language of the hyperstructure theory this can be described as constructing a hyperoperation on a set H endowed with a binary operation $\cdot$ and a relation $\leq$. The special case of such hyperstructures, when $(H, \cdot, \leq)$ is a quasi-ordered semigroup and the hyperoperation $* : H \times H \to \mathcal{P}^*(H)$ is defined by

$$a * b = \{x \in H; a \cdot b \leq x\} \tag{1}$$

for all $(a, b) \in H^2$, is known as *EL–hyperstructures*. These had been constructed by a number of authors including Borzooei, et al., Chvalina, Davvaz,

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Dehghan Nezhad, Hošková and others [1, 3, 9, 11] (see also book [8], sections 8.3 and 8.4.) before they were studied from the theoretical point of view [20, 21, 22].

When one views the construction (1) from the perspective of the lattice theory, it becomes obvious that dualizing the concept, involving one more "similar" hyperoperation or changing $a \cdot b \leq x$ in (1) to the interval definition using two hyperoperations may lead to interesting results. Moreover, despite all the advances of the hyperstructure theory, the concept of a *matrix*, i.e. a two-dimensional scheme of $m \times n$ entries, has been studied in it only occassionaly. The exception to this rule is the concept of H_v*–matrices* used by Vougiouklis in the representation theory [25, 26, 27, 28]. However, since entries of H_v–matrices are elements of H_v*–rings*, i.e. hyperstructures with two hyperoperations, defining and working with H_v–matrix multiplication, trace or rank of H_v–matrices or other matrical concepts is complicated or yet unexplored. In fact, when in his overview paper [25], Vougiouklis presents "some of the open problems arising on the topic in the procedure to find representations on hypergroups", four out of the eight presented problems regard H_v–matrices.

Therefore, we study the concept inspired by (1) on the sets of matrices. We show that our considerations naturally result in H_v–matrices which might provide a tool for better applications of this concept. Also, we show that in the context of lattices, there exist other classes of hyperstructures analogous to EL–hyperstructures. In this we expand results of Davvaz, Leoreanu-Fotea, Rosenberg or Varlet [13, 16, 17, 24].

Notice that the concept of *ordering* has been connected to the hyperstructure theory from its very beginnings. Our paper falls within the area of *hyperstructures constructed from ordered (semi)groups*. Apart from this *hyperstructures constructed from ordered sets* (such as *quasi-order hypergroups*) as well as *ordered hyperstructures* have been studied extensively. For details see e.g. [2, 5, 6] or papers initiated by the introduction of *ordered hyperstructures* in [10]. Moreover, some of these concepts have been studied in the n–ary context as well [7, 14]. A short overview of possible approaches can be found in e.g. [20].

2 Theory

2.1 Basic concepts and terminology

In this paper we use concepts of the *theory of lattices* and of the *hyperstructure theory*. Since the basic definitions of the lattice theory are well known, we recall some definitions of the hyperstructure theory only. By a *binary hyperoperation* $*$ we mean a mapping $* : H \times H \to \mathcal{P}^*(H)$, where H is a non-empty set and $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H. By a *semihypergroup* we mean a hypergroupoid $(H, *)$ such that, for all $(a, b, c) \in H^3$, there is $(a * b) * c = a * (b * c)$. If instead of equality we require non-empty intersection only, we talk about an *H_v-semigroup.* A *quasihypergroup* is a hypergroupoid $(H, *)$ which satisfies the reproduction law, i.e. that, for all $a \in H$, there is $H * a = a * H$. If the reproduction law holds in H_v-semigroups, we call such a hyperstructure an *H_v-group.* If two hyperoperations are defined on H such that $(H, *)$ is an H_v-group and $(H, \star)$ is an H_v-semigroup, and $\star$ is weakly distributive over $*$ from both left and right, i.e. equality is replaced with non-empty intersection in the distributive laws, then $(H, *, \star)$ is called an *H_v–ring.* A matrix, entries of which are elements of H_v–rings are called *H_v–matrices.* Finally, by a *join space* we mean a commutative hypergroup $(H, *)$ such that for all $(a, b, c, d) \in H^4$ the following implication holds: $a/b \approx c/d \Rightarrow a * d \approx b * c$. Here, $a/b = \{x \in H; a \in x * b\}$ and $\approx$ denotes non-empty intersection. For further introduction to the hyperstructure theory see canonical books [5, 6, 8, 26] or paper [12].

2.2 Context and notation

We denote $\mathbb{M}_{m,n}(\mathcal{S})$ the set of all $m \times n$ matrices with entries from a suitable set $\mathcal{S}$, i.e.

$$\mathbb{M}_{m,n}(\mathcal{S}) = \{\mathbf{M} = [m_{i,j}]); m_{i,j} \in \mathcal{S}, i = \{1, \ldots, m\}, j = \{1, \ldots, n\}\}. \tag{2}$$

On $\mathbb{M}_{m,n}(\mathcal{S})$ we, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$, naturally define relation $\leq_M$ in an entry-wise manner by

$$\mathbf{A} \leq_M \mathbf{B} \text{ if } a_{i,j} \leq_e b_{i,j} \text{ for all } i = \{1, \ldots, m\}, j = \{1, \ldots, n\}, \tag{3}$$

where $\leq_e$ is a suitable relation between entries of the matrices. Suppose that $(\mathcal{S}, \inf, \sup, \leq_e)$ is a lattice and define the *minimum of matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$ by

$$\min\{\mathbf{A}, \mathbf{B}\} = \mathbf{C}, \text{ where } \mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}) \text{ is such that } c_{i,j} = \inf\{a_{i,j}, b_{i,j}\} \tag{4}$$

for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$, in case of two matrices and analogically in case of more matrices; and the *maximum of matrices* $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$ by

$$\max\{\mathbf{A}, \mathbf{B}\} = \mathbf{D}, \text{ where } \mathbf{D} \in \mathbb{M}_{m,n}(\mathcal{S}) \text{ is such that } d_{i,j} = \sup\{a_{i,j}, b_{i,j}\} \tag{5}$$

for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$, in case of two matrices and analogically in case of more matrices.

We will show later on that the straightforwardness and suspected “simplicity” of the above definitions is in fact their advantage.

2.3 Relation to EL–hyperstructures

Following is the main result concerning construction (1).

Lemma 1. *([2], Theorem 1.3, p. 146) Let $(H, \cdot, \leq)$ be a partially ordered semigroup. Binary hyperoperation $* : H \times H \to \mathcal{P}^*(H)$ defined by*

$$a * b = \{x \in H; a \cdot b \leq x\} \tag{6}$$

is associative. The semihypergroup $(H, *)$ is commutative if and only if the semigroup $(H, \cdot)$ is commutative.

If we suppose that $(\mathcal{S}, \inf, \sup, \leq_e)$ is a *lattice*, then the following obviously holds concerning relation (3) and operations (4) and (5) defined on $\mathbb{M}_{m,n}(\mathcal{S})$.

Lemma 2. *The operations* min *and* max *defined on $\mathbb{M}_{m,n}(\mathcal{S})$ by (4) and (5) respectively, are idempotent, commutative and associative. $(\mathbb{M}_{m,n}(\mathcal{S}), \leq_M)$ is a partially ordered set. $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \leq_M)$, and $(\mathbb{M}_{m,n}(\mathcal{S}), \max, \leq_M)$, are partially ordered semigroups.*

3 Results

3.1 Min-max hyperstructures of matrices

Lemma 2 allows us to make an immediate conclusion regarding the structure $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max, \leq_M)$.

Theorem 1. *$(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max, \leq_M)$, where operations* min *and* max *are defined by (4) and (5) and $\leq_M$ is defined by (3), is a lattice.*

Proof. Lemma 2 verifies commutativity, associativity and idempotency. The absorption laws hold thanks to the relationship between $\leq_M$ and $\leq_e$, expressed by (3), and the fact that $(\mathcal{S}, \inf, \sup, \leq_e)$ is a lattice. $\square$

Now that we have established the context of $\mathbb{M}_{m,n}(\mathcal{S})$, we define two pairs of dual hyperoperations on $\mathbb{M}_{m,n}(\mathcal{S})$ using (3) and (4), or (5), respectively.

First, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$ we define

$$\mathbf{A} \circ \mathbf{B} = \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}); \min\{\mathbf{A}, \mathbf{B}\} \leq_M \mathbf{C}\}, \tag{7}$$

i.e., for all $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \circ \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} =$$
$$\left\{ \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \dots & \dots & \dots \\ c_{m1} & \dots & c_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}); \inf\{a_{ij}, b_{ij}\} \leq_e c_{ij} \right\}$$

and dually

$$\mathbf{A} \bullet \mathbf{B} = \{\mathbf{D} \in \mathbb{M}_{m,n}(\mathcal{S}); \max\{\mathbf{A}, \mathbf{B}\} \geq_M \mathbf{D}\}, \tag{8}$$

i.e., for all $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$,

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \bullet \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mn} \end{bmatrix} =$$
$$\left\{ \begin{bmatrix} d_{11} & \dots & d_{1n} \\ \dots & \dots & \dots \\ d_{m1} & \dots & d_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}); \sup\{a_{ij}, b_{ij}\} \geq_e d_{ij} \right\}.$$

Lemma 3. *For an arbitrary quadruple* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \in \mathbb{M}_{m,n}(\mathcal{S})$ *we have* $\mathbf{A}_1 \circ \mathbf{A}_2 \approx \mathbf{A}_3 \circ \mathbf{A}_4$ *and* $\mathbf{A}_1 \bullet \mathbf{A}_2 \approx \mathbf{A}_3 \bullet \mathbf{A}_4$.

Proof. The proof for both hyperoperations is analogous, we include it only for hyperoperation $\circ$. Suppose $\mathbf{A}_i, i \in \{1, 2, 3, 4\}$, are arbitrary elements of $\mathbb{M}_{m,n}(\mathcal{S})$. Denote $\mathbf{B} = \max\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$. Since $\mathbb{M}_{m,n}(\mathcal{S})$ is a lattice, there is $\mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$. Moreover, there is $\min\{\mathbf{A}_1, \mathbf{A}_2\} \leq_M \mathbf{B}$ and $\min\{\mathbf{A}_3, \mathbf{A}_4\} \leq_M \mathbf{B}$. As a result, $\mathbf{B} \in \mathbf{A}_1 \circ \mathbf{A}_2$ and also $\mathbf{B} \in \mathbf{A}_3 \circ \mathbf{A}_4$, which proves the lemma. □

Example 1. *Let* $\mathcal{S}$ *be the lattice of divisors of a suitable natural number* n *with* $\inf\{a, b\}$ *being the greatest common divisor of* $a, b \in \mathbb{N}$, $\sup\{a, b\}$ *being the least common multiple of* a, b *and* $a \leq_e b$ *if* $a|b$. *For e.g.* $n = 120$, *divisors of which are* $1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, 120$, *construct* $\mathbb{M}_{2,2}(\mathcal{S})$ *and regard an arbitrary quadruple of matrices* $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4 \in \mathbb{M}_{2,2}(\mathcal{S})$, *e.g.* $\mathbf{A}_1 = \begin{bmatrix} 8 & 15 \\ 3 & 6 \end{bmatrix}$, $\mathbf{A}_2 = \begin{bmatrix} 10 & 12 \\ 20 & 24 \end{bmatrix}$ $\mathbf{A}_3 = \begin{bmatrix} 1 & 2 \\ 5 & 3 \end{bmatrix}$, $\mathbf{A}_4 = \begin{bmatrix} 8 & 12 \\ 30 & 1 \end{bmatrix}$. *Then* $\mathbf{B} =$

$\begin{bmatrix} 40 & 60 \\ 60 & 24 \end{bmatrix}$,

$$\mathbf{A}_1 \circ \mathbf{A}_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 2|a_{11}, 3|a_{12}, 1|a_{21}, 6|a_{22} \right\},$$
$$\mathbf{A}_3 \circ \mathbf{A}_4 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 1|a_{11}, 2|a_{12}, 5|a_{21}, 1|a_{22} \right\},$$

and obviously $\mathbf{B} \in \mathbf{A}_1 \circ \mathbf{A}_2 \cap \mathbf{A}_3 \circ \mathbf{A}_4$.

Theorem 2. $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ *and* $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet)$ *are join spaces.*

Proof. Since the hyperoperations $\circ$ and $\bullet$ are dual, i.e. the respective proofs would be analogous, we will prove only the fact that $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ is a join space. First of all, commutativity of the hyperoperation is obvious. Next, thanks to Lemma 1 we immediately get that $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ is a semihypergroup.

Reproduction law, i.e. condition $\mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S}) = \mathbb{M}_{m,n}(\mathcal{S})$ holds for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$: It is evident that $\mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S}) \subseteq \mathbb{M}_{m,n}(\mathcal{S})$, for any $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$. As far as the opposite inclusion, i.e. $\mathbb{M}_{m,n}(\mathcal{S}) \subseteq \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$, for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$, is concerned, notice that

$$\mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S}) = \bigcup_{\mathbf{X} \in \mathbb{M}_{m,n}(\mathcal{S})} \mathbf{A} \circ \mathbf{X} = \bigcup_{\mathbf{X} \in \mathbb{M}_{m,n}(\mathcal{S})} \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}); \min\{\mathbf{A}, \mathbf{X}\} \leq C\}.$$

For a fixed $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$ and an arbitrary $\mathbf{M} \in \mathbb{M}_{m,n}(\mathcal{S})$ the following cases are possible:

1. If $\mathbf{M} \leq_M \mathbf{A}$, then $\min\{\mathbf{A}, \mathbf{M}\} = \mathbf{M}$ and since $\leq_M$ is reflexive, there is $\mathbf{M} \in \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$.
2. If $\mathbf{A} \leq_M \mathbf{M}$, then $\min\{\mathbf{A}, \mathbf{M}\} = \mathbf{A}$ which means that $\mathbf{M} \in \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$.
3. If $\mathbf{A}$ and $\mathbf{M}$ are not in relation $\leq_M$, then there is $\min\{\mathbf{A}, \mathbf{M}\} \leq \mathbf{M}$, which means that $\mathbf{M} \in \mathbf{A} \circ \mathbb{M}_{m,n}(\mathcal{S})$.

Therefore, $(\mathbb{M}, \circ)$ is a commutative hypergroup. Finally, the transposition axiom holds thanks to Lemma 3. □

Remark 1. *Notice that Račková in [23] includes a proof of the fact that some EL–hyperstructures are join spaces. However, the assumption of her theorem is that the single-valued structure, in our case* $(\mathbb{M}_{m,n}(\mathcal{S}), \inf, \leq_e)$*, or* $(\mathbb{M}_{m,n}(\mathcal{S}), \sup, \leq_e)$*, is a partially ordered group. Therefore, the result obtained in [23] could not have been applied, since* $(\mathbb{M}_{m,n}(\mathcal{S}), \inf)$*,* $(\mathbb{M}_{m,n}(\mathcal{S}), \sup)$ *are semigroups only. For a deeper insight in this issue cf. [21]; for a corollary see Section 3.4.*

Now, analogous to (7) and (8) we for matrices $\mathbf{A},\mathbf{B}\in\mathbb{M}_{m,n}(\mathcal{S})$ define

$$\mathbf{A}*\mathbf{B}=\{\mathbf{C}\in\mathbb{M}_{m,n}(\mathcal{S});\max\{\mathbf{A},\mathbf{B}\}\leq_M\mathbf{C}\},\tag{9}$$

i.e., for all $i\in\{1,\dots,m\}, j\in\{1,\dots,n\}$,

$$\begin{bmatrix}a_{11} & \dots & a_{1n}\\ \dots & \dots & \dots\\ a_{m1} & \dots & a_{mn}\end{bmatrix}*\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \dots & \dots & \dots\\ b_{m1} & \dots & b_{mn}\end{bmatrix}=$$
$$\left\{\begin{bmatrix}c_{11} & \dots & c_{1n}\\ \dots & \dots & \dots\\ c_{m1} & \dots & c_{mn}\end{bmatrix}\in\mathbb{M}_{m,n}(\mathcal{S});\sup\{a_{ij},b_{ij}\}\leq_e c_{ij}\right\}.$$

and dually

$$\mathbf{A}\star\mathbf{B}=\{\mathbf{D}\in\mathbb{M}_{m,n}(\mathcal{S});\min\{\mathbf{A},\mathbf{B}\}\geq_M\mathbf{D}\},\tag{10}$$

i.e., for all $i\in\{1,\dots,m\}, j\in\{1,\dots,n\}$,

$$\begin{bmatrix}a_{11} & \dots & a_{1n}\\ \dots & \dots & \dots\\ a_{m1} & \dots & a_{mn}\end{bmatrix}\star\begin{bmatrix}b_{11} & \dots & b_{1n}\\ \dots & \dots & \dots\\ b_{m1} & \dots & b_{mn}\end{bmatrix}=$$
$$\left\{\begin{bmatrix}d_{11} & \dots & d_{1n}\\ \dots & \dots & \dots\\ d_{m1} & \dots & d_{mn}\end{bmatrix}\in\mathbb{M}_{m,n}(\mathcal{S});\inf\{a_{ij},b_{ij}\}\geq_e d_{ij}\right\}.$$

Example 2. *Suppose that $\mathcal{S}$ is a lattice of non-negative integer pairs, where we set $(a,b)\leq_e(c,d)$ if $a\leq c$ and $b\leq d$, and consider 2×2 matrices of such entries, e.g.* $\mathbf{A}=\begin{bmatrix}(5,8) & (3,0)\\ (2,4) & (1,9)\end{bmatrix}$ *and* $\mathbf{B}=\begin{bmatrix}(7,2) & (2,1)\\ (6,1) & (3,5)\end{bmatrix}$. *Then the hyperproduct* $\mathbf{A}*\mathbf{B}$ *is*

$$\mathbf{A}*\mathbf{B}=\left\{\begin{bmatrix}(a_{11}^1,a_{11}^2) & (a_{12}^1,a_{12}^2)\\ (a_{21}^1,a_{21}^2) & (a_{22}^1,a_{22}^2)\end{bmatrix}\in\mathbb{M}_{2,2}(\mathcal{S})\right\},$$

where the entries are such that

$$7\leq a_{11}^1, 8\leq a_{11}^2, 3\leq a_{12}^1, 1\leq a_{12}^2, 6\leq a_{21}^1, 4\leq a_{21}^2, 3\leq a_{22}^1, 9\leq a_{22}^2.$$

Theorem 3. $(\mathbb{M}_{m,n}(\mathcal{S}),*)$ *and* $(\mathbb{M}_{m,n}(\mathcal{S}),\star)$ *are commutative semihypergroups.*

Proof. For hyperoperation "$*$" follows directly from Lemma 1 and Lemma 2; for hyperoperation "$\star$" follows from the fact that min and max are dual. $\square$

Remark 2. *The above semihypergroups do not satisfy the reproduction axiom. Since the hyperoperations $\star$ and $*$, or rather operations* min *and* max*, are dual, we will demonstrate this on $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ only and show that the condition $\mathbf{A} * \mathbb{M}_{m,n}(\mathcal{S}) \neq \mathbb{M}_{m,n}(\mathcal{S})$ does not hold for all $\mathbf{A} \in \mathbb{M}_{m,n}(\mathcal{S})$. Notice that*

$$\mathbf{A}*\mathbb{M}_{m,n}(\mathcal{S}) = \bigcup_{\mathbf{X}\in\mathbb{M}_{m,n}(\mathcal{S})} \mathbf{A}*\mathbf{X} = \bigcup_{\mathbf{X}\in\mathbb{M}_{m,n}(\mathcal{S})} \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}); \max\{\mathbf{A}, \mathbf{X}\} \leq_M C\}.$$

Now, for matrices $\mathbf{A}, \mathbf{X} \in \mathbb{M}_{m,n}(\mathcal{S})$ such that $\mathbf{X} \leq_M \mathbf{A}$ there is $\max\{\mathbf{A}, \mathbf{X}\} = \mathbf{A}$ and if we take $\mathbf{M} \in \mathbb{M}_{m,n}(\mathcal{S})$ such that $M \leq_M \mathbf{A}$, then $\mathbf{M} \notin \mathbf{A} * \mathbb{M}_{m,n}(\mathcal{S})$.

Theorem 4. *The transposition axiom holds in $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \star)$.*

Proof. Once again, it is sufficient to prove the statement for $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ only. The proof is analogous to the proof of Lemma 3, only for the matrix $\mathbf{B} = \max\{\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4\}$ there holds $\mathbf{B} \geq_M \max\{\mathbf{A}_1, \mathbf{A}_4\}$ and $\mathbf{B} \geq_M \max\{\mathbf{A}_2, \mathbf{A}_3\}$, i.e. $\mathbf{B} \in \mathbf{A}_1 * \mathbf{A}_4$ and simultaneously $\mathbf{B} \in \mathbf{A}_2 * \mathbf{A}_3$. $\square$

Example 3. *If in Example 1 we use "$*$" instead of "$\circ$", we get that*

$$\mathbf{A}_1 * \mathbf{A}_2 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 40|a_{11}, 60|a_{12}, 60|a_{21}, 24|a_{22} \right\},$$
$$\mathbf{A}_3 * \mathbf{A}_4 = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, 8|a_{11}, 12|a_{12}, 30|a_{21}, 3|a_{22} \right\},$$

and obviously $\mathbf{B} = \begin{bmatrix} 40 & 60 \\ 60 & 24 \end{bmatrix} \in \mathbf{A}_1 \circ \mathbf{A}_2 \cap \mathbf{A}_3 \circ \mathbf{A}_4$.

Remark 3. *Notice that even though the transposition axiom is usually studied in hypergroups, its validity is neither restricted to nor follows from the validity of the reproductive law. Transposition axiom in semihypergroups which are not hypergroups has been studied e.g. by Massouros and Massouros [18].*

Among the very basic notions of the hyperstructure theory there is the idea of proclaiming the line segment as the result of the hyperoperation applied on its endpoints. Inspired by this, for an arbitrary pair of matrices $\mathbf{A}, \mathbf{B} \in \mathbb{M}_{m,n}(\mathcal{S})$, we define

$$\mathbf{A} \odot \mathbf{B} = \{\mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S}); \min\{\mathbf{A}, \mathbf{B}\} \leq_M \mathbf{C} \leq_M \max\{\mathbf{A}, \mathbf{B}\}\}, \tag{11}$$

i.e., for all $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$,

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} \odot \begin{bmatrix} b_{11} & \ldots & b_{1n} \\ \ldots & \ldots & \ldots \\ b_{m1} & \ldots & b_{mn} \end{bmatrix} =$$
$$\left\{ \begin{bmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{m1} & \ldots & c_{mn} \end{bmatrix} \in \mathbb{M}_{m,n}(\mathcal{S}); \inf\{a_{ij}, b_{ij}\} \leq_e c_{ij} \leq_e \sup\{a_{ij}, b_{ij}\} \right\}.$$

Notice that (11) is in fact a matrix variation of a hyperoperation defined by Varlet [24], Definition 1, which is frequentely used in machine learning applications and is often studied alongside with another hyperoperation introduced by Nakano [19] and studied e.g. by Comer [4], which creates join spaces from modular lattices. Varlet's ideas have been studied and used by e.g. Davvaz, Leoreanu-Fotea or Rosenberg [13, 15, 16, 17]. The nature of $(\mathbb{M}_{m,n}(\mathcal{S}), \odot)$ can be easily established with the help of results obtained by Varlet [24].

In this respect, first recall that a lattice $(\mathfrak{L}, \wedge, \vee)$ such that $\wedge$ distributes over $\vee$ (and dually $\vee$ over $\wedge$) is called *distributive*.

Theorem 5. *The lattice* $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ *is distributive if and only if the lattice* $(\mathcal{S}, \inf, \sup)$ *is distributive.*

Proof. The proof is rather obvious thanks to the straightforward correspondence between relations "$\leq_M$" and "$\leq_e$" suggested by (3) and correspondence between the definition of minimum and maximum of matrices using infima and suprema of their entries. If $(\mathcal{S}, \inf, \sup)$ is distributive, then distributive laws are valid for all $a_{ij}, b_{ij}, c_{ij} \in \mathcal{S}$, i.e. distributive laws are valid for matrices as well, which means that $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ is distributive. On the other hand, if $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ is distributive, then $\max\{\mathbf{A}, \min\{\mathbf{B}, \mathbf{C}\}\} = \min\{\max\{\mathbf{A}, \mathbf{B}\}, \max\{\mathbf{B}, \mathbf{C}\}\}$ for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{M}_{m,n}(\mathcal{S})$ and thanks to the definition of the minimum and maximum of matrices we immediately have that $(\mathcal{S}, \inf, \sup)$ is distributive. $\square$

Definition 1. [24] *Let* $\mathfrak{L}_{\leq} = (L, \wedge, \vee)$ *be a lattice with join* $\wedge$*, meet* $\vee$ *and order relation* $\leq$ *and let:*

$$\forall (a, b) \in L^2, a \diamond b = \{x \in L \mid a \wedge b \leq x \leq a \vee b\}.$$

Theorem 6. [24] *For a lattice* $\mathfrak{L}_{\leq}$ *the following are equivalent:*

(1) $\mathfrak{L}_{\leq}$ *is distributive,*

(2) $\mathbb{L}_{\leq} = (L, \diamond)$ *is join space.*

Theorem 5 and Varlet's results allow us to immediately state the following.

Corolary 1. $(\mathbb{M}_{m,n}(\mathcal{S}), \odot)$ *is a join space if and only if the lattice* $(\mathcal{S}, \inf, \sup)$ *is distributive.*

3.2 H_v–matrices

Constructions of Section 3.1 naturally result in H_v–rings, i.e. as a consequence in H_v–matrices. In Theorem 5 we have already seen that sets of matrices $(\mathbb{M}_{m,n}(\mathcal{S}), \min, \max)$ are distributive lattices if and only if sets $(\mathcal{S}, \inf, \sup)$ of their entries are distributive lattices. Moreover, the following – stronger – lemma holds.

Lemma 4. *Let* $(H, +, \leq)$, $(H, \cdot, \leq)$ *be partially ordered semigroups such that "$\cdot$" distributes over "$+$" from both left and right. For all* $a, b \in H$ *define*

$$\begin{aligned} a +_{\leq} b &= \{x \in H; a + b \leq c\}, \\ a \cdot_{\leq} b &= \{y \in H; a \cdot b \leq c\}. \end{aligned}$$

Then "$\cdot_{\leq}$" weakly distributes over "$+_{\leq}$" from both sides.

Proof. For an arbitraty $h \in H$ we will denote the set $\{x \in H; h \leq x\}$ by $[h)_{\leq}$. Now, thanks to Lemma 1 both $(H, \cdot_{\leq})$ and $(H, +_{\leq})$ are semihypergroups. For arbitrary $a, b, c \in H$ consider the element $a \cdot (b + c)$ which, thanks to distributivity, equals $a \cdot b + a \cdot c$. Notice that

$$a \cdot_{\leq} (b +_{\leq} c) = a \cdot_{\leq} [b + c)_{\leq} = \bigcup_{x \in [b+c)_{\leq}} a \cdot x = \bigcup_{b+c \leq x} a \cdot x$$

and on the other hand

$$(a \cdot_{\leq} b) +_{\leq} (a \cdot_{\leq} c) = [a \cdot b)_{\leq} +_{\leq} [a \cdot c)_{\leq} = \bigcup_{y \in [a \cdot b)_{\leq}, z \in [a \cdot c)_{\leq}} [y + z)_{\leq} = \bigcup_{a \cdot b \leq y, a \cdot c \leq z} [y + z)_{\leq}$$

and since the relation $\leq$ is reflexive, we immediately see that $a \cdot (b + c) = a \cdot b + a \cdot c$ is the common element of both regarded sets. Analogous reasoning can be done for the element $(a + b) \cdot c = a \cdot c + b \cdot c$. □

Therefore, we straightforwardly get the following:

Theorem 7. *If the lattice* $(\mathcal{S}, \inf, \sup)$ *is distributive, then* $(\mathbb{M}_{m,n}(\mathcal{S}), \circ, *)$ *and* $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet, \star)$ *are H_v–rings.*

Proof. Follows immediately from Theorem 2, which provides the hypergroups, Theorem 3, which provides the semihypergroups, and Lemma 4, which provides weak distributivity (because $\mathcal{S}$ is distributive). □

Corolary 2. *If $\mathcal{S}$ is a distributive lattice, then $\mathbb{M}_{m,n}(\mathcal{S})$ is the set of H_v–matrices.*

Proof. If in Theorem 7 we set $m = n = 1$, then $\mathbb{M}_{m,n}(\mathcal{S})$ becomes $\mathcal{S}$, i.e. the set from which we take entries of $\mathbb{M}_{m,n}(\mathcal{S})$. Definitions of hyperoperations $\circ, \bullet, *, \star$ simplify accordingly. $\square$

Thus we see that using Theorem 7 and Corollary 2 we can in fact construct H_v–matrices of different classes: first of all, matrices, entries of which are elements of $\mathcal{S}$ (because, thanks to Corollary 2, $\mathcal{S}$ is an H_v–ring). We have denoted this set of matrices by $\mathbb{M}_{m,n}(\mathcal{S})$. Yet since $\mathbb{M}_{m,n}(\mathcal{S})$ is itself a lattice, which is distributive if and only if $\mathcal{S}$ is distributive, we can apply Theorem 7 and regard elements of $\mathbb{M}_{m,n}(\mathcal{S})$ as entries of matrices again. For easier future reference we can denote this set of matrices as $\mathbb{M}^2_{m,n}(\mathcal{S})$.

Remark 4. *Notice that the conditions of definition of H_v–ring are rather weak for our context. All our hyperoperations are not only weakly associative but associative. Also, we do not obtain H_v-groups but hypergroups. In other words, three things remain to be secured for $(\mathbb{M}_{m,n}(\mathcal{S}), \circ, *)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet, \star)$ to become* Krasner hyperrings*: existence of a scalar identity of $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ or $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet)$, existence of absorbing elements of $(\mathbb{M}_{m,n}(\mathcal{S}), *)$ or $(\mathbb{M}_{m,n}(\mathcal{S}), \star)$, and distributivity of the hyperoperations instead of weak distributivity shown by Lemma 4. However, Theorem 7 of [22] shows that the existence of scalar identities of $(\mathbb{M}_{m,n}(\mathcal{S}), \circ)$ and $(\mathbb{M}_{m,n}(\mathcal{S}), \bullet)$ is not possible.*

3.3 Properties of the hyperstructures

Some of the hyperstructures, which have been constructed in this paper, are EL–hyperstructures. Some of the hyperstructures are join spaces (not necessarily EL–join spaces) while others are EL–hypergroups which are not constructed from partially ordered groups. Properties of all these types of hyperstructures can be derived with the help of papers [12, 20, 21, 22]. For a deeper insight in the properties of join spaces of the same type as $(\mathbb{M}_{m,n}(\mathcal{S}), \odot)$, see [6], chapter 4, or [17].

3.4 The special case of $m = n = 1$

When setting $m = n = 1$ in $\mathbb{M}_{m,n}(\mathcal{S})$, we obtain the original lattice $\mathcal{S}$. This enables us to construct analogies of EL–hyperstructures (6). For all

$(a, b, c, d) \in H^4$, hyperoperations (7), (8), (9), (10), (11) in this case reduce to

$$a \circ b = \{c \in \mathcal{S}; \inf\{a, b\} \leq_e c\}$$
$$a \bullet b = \{d \in \mathcal{S}; \sup\{a, b\} \geq_e c\}$$
$$a * b = \{c \in \mathcal{S}; \sup\{a, b\} \leq_e c\}$$
$$a \star b = \{d \in \mathcal{S}; \inf\{a, b\} \geq_e d\}$$
$$a \odot b = \{c \in \mathcal{S}; \inf\{a, b\} \leq_e c \leq_e \sup\{a, b\}\}$$

and we immediately get the following corollary of results of Section 3.1.

Corolary 3. *If* $(\mathcal{S}, \inf, \sup, \leq_e)$ *is a lattice, then*

1. $(\mathcal{S}, \circ)$ *and* $(\mathcal{S}, \bullet)$ *are join spaces,*
2. $(\mathcal{S}, *)$ *and* $(\mathcal{S}, \star)$ *are semihypergroups which are not hypergroups yet satisfy the transposition axiom.*

Thus – in lattices – the concept of EL–hyperstructures can be not only dualized but also its natural analogy can be proved. As we have already seen, in distributive lattices for hyperoperation $\odot$, the considerations suggested by the concept of EL–hyperstructures (6) are linked to classical results [24].

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