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On the weak limit of compact operators on the reproducing kernel Hilbert space and related questions

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Abstract

By applying the so-called Berezin symbols method we prove a Gohberg-Krein type theorem on the weak limit of compact operators on the non-standard reproducing kernel Hilbert space which essentially improves the similar results of Karaev [5]. We also in terms of reproducing kernels and Berezin symbols investigate the structure of invariant subspaces of compact operators.

1 Introduction

It is a classical result that, if a sequence (K_n) of compact operators on a Hilbert space (or, in Banach space) uniformly converges to an operator K, then K is also compact. Also it is well known that neither strong limit nor weak limit of compact operators is compact in general. Thus the following question is essential (see Karaev [5]): under which conditions is the weak limit of a sequence of compact operators, a compact operator too? A first answer to this was provided in Gohberg and Krein [4, Chapter 3]. Namely, the authors proved that, under some growth conditions on the singular numbers (s-numbers) of compact operators K_n, the weak limit of (K_n) is compact. Further, this result of Gohberg and Krein has been somewhat improved by

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Pehlivan and Karaev [9] by using the concept of statistical convergence of the summability theory. Recently, Karaev [5] proved the following Gohberg - Krein type theorem by applying Berezin symbols technique.

Theorem 1. *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a standard functional Hilbert space on some set Ω and (K_n) be a sequence of compact operators on $\mathcal{H}$ weakly converging to an operator A on $\mathcal{H}$. If the double limit*

$$\lim_{\lambda \to \partial\Omega,\ n \to \infty} \widetilde{U^{-1} K_n U}(\lambda) = L$$

exists for every unitary operator $U : \mathcal{H} \to \mathcal{H}$, then A is compact operator and $L = 0$ holds.

The proof of this theorem essentially uses a results of Nordgren and Rosenthal [8, Corollary 2.8] which characterize compact operators on the standard reproducing kernel Hilbert spaces in terms of Berezin symbols their unitary orbits.

In this article we will prove that the hypothesis of standardness of the reproducing kernel Hilbert space (or, equivalently, functional Hilbert space) $\mathcal{H}$ in Theorem 1 can be highly weakened. We also investigate in terms of reproducing kernels and Berezin symbols the invariant subspaces of compact operators on the reproducing kernel Hilbert spaces.

Before giving our results we need some necessary notations and preliminaries.

2 Notations

Let Ω be a subset of a topological space X such that the boundary $\partial\Omega$ is nonempty. Let $\mathcal{H} = \mathcal{H}(\Omega)$ be an infinite dimensional Hilbert space of complex-valued functions defined on Ω. We say that (see, for instance, Aronzajn [1], Karaev [6] and Saitoh [10, 11]) $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS) if the following two conditions are satisfied:

(**i**) for any $\lambda \in \Omega$, the evaluating functionals $f \to f(\lambda)$ are continuous on $\mathcal{H}$;

(**ii**) for any $\lambda \in \Omega$, there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$.

By the classical Riesz representation theorem, the assumption (i) implies that for any $\lambda \in \Omega$ there exists $k_{\mathcal{H},\lambda} \in \mathcal{H}$ such that $f(\lambda) = \langle f, k_{\mathcal{H},\lambda} \rangle_{\mathcal{H}}$ for any $f \in \mathcal{H}$. This function $k_{\mathcal{H},\lambda}$ is called the reproducing kernel of $\mathcal{H}$ at point λ. On the other hand, by the assumption (ii), we have $k_\lambda \neq 0$ and we denote by $\widehat{k}_{\mathcal{H},\lambda}$ the normalized reproducing kernel, that is

$$\widehat{k}_{\mathcal{H},\lambda} = \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}}, \quad \lambda \in \Omega.$$

If $\mathcal{B}(\mathcal{H})$ denotes the Banach algebra of linear and bounded operators acting in $\mathcal{H} = \mathcal{H}(\Omega)$, then the Berezing symbol $\widetilde{A}$ of any operator $A \in \mathcal{B}(\mathcal{H})$ is the complex valued function defined on Ω by

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle, \quad \lambda \in \Omega.$$

Clearly, $\widetilde{A}$ is bounded on Ω and

$$\sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right| \leq \|A\|.$$

Following Nordgren and Rosenthal [8], we say that a RKHS $\mathcal{H}$ is standard, if $\widehat{k}_{\mathcal{H},\lambda} \to 0$ weakly as $\lambda \to \xi$, for any point $\xi \in \partial\Omega$. Most of RKHS, including the Hardy and Bergman Hilbert spaces of analytic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ are standard in this sense.

For $\mathcal{H} = \mathcal{H}(\Omega)$ a standard RKHS and K a compact operator on $\mathcal{H}$, obviously $\lim_{n\to\infty} \widetilde{K}(\lambda_n) = 0$ whenever $\{\lambda_n\} \subset \Omega$ converges to a point of $\partial\Omega$ (because compact operators send weakly convergent sequences into strongly convergent ones and $\left|\widetilde{K}(\lambda)\right| \leq \left\|K\widehat{k}_{\mathcal{H},\lambda}\right\|$ according to Cauchy-Schwartz inequality). In this sense, the Brezin symbol of compact operator on a standard functional Hilbert space vanishes on the boundary.

3 Berezin symbols and weak limit of compact operators

In this section, we show that the hypothesis of standardness of the RKHS $\mathcal{H}$ in Theorem 1, which due to Karaev [5, Theorem 4.1], can be essentially weakened. For this purpose, following Chalendar et al. [3], for any RKHS $\mathcal{H} = \mathcal{H}(\Omega)$ (not necessarily standard), denote by $\partial_{\mathcal{H}}\Omega$ the subset of the boundary of the set Ω defined by

$$\partial_{\mathcal{H}}\Omega = \left\{ \xi \in \partial\Omega : \widehat{k}_{\mathcal{H},\lambda} \to 0 \text{ weakly whenever } \lambda \to \xi \right\}.$$

Clearly, $\mathcal{H}$ is standard if and only if $\partial_{\mathcal{H}}\Omega = \partial\Omega$. In the case where $\partial_{\mathcal{H}}\Omega \neq \emptyset$, the authors of the paper [3] proved the following analogue of the results of Nordgren and Rosenthal [8, Corollary 2.8] mentioned in Section 1.

Theorem 2. *([3]). Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS on Ω such that $\partial_{\mathcal{H}}\Omega \neq \emptyset$, and let $A \in \mathcal{B}(\mathcal{H})$. Then the following assertions are equivalent.*

(i) *A is compact;*

(ii) *for every point $\xi \in \partial_{\mathcal{H}}\Omega$ and every unitary operator U on $\mathcal{H}$, we have $\lim_{\lambda\to\xi} \widetilde{U^{-1}AU}(\lambda) = 0$;*

(**iii**) *there exists a sequence* $\{\lambda_n\}_{n\geq 1}$ *of points in* Ω, *converging to a point* $\xi \in \partial_{\mathcal{H}}\Omega$, *such that for every unitary operator* U *on* $\mathcal{H}$, *we have* $\lim_{n\to+\infty} \widetilde{U^{-1}AU}(\lambda_n) = 0$.

Here we will use Theorem 2 to prove the following main result of the present section.

Theorem 3. *Let* $\mathcal{H} = \mathcal{H}(\Omega)$ *be a RKHS on* Ω *such that* $\partial_{\mathcal{H}}\Omega \neq \emptyset$, *and* $(K_n)_{n\geq 1}$ *be a sequence of compact operators on* $\mathcal{H}$ *weakly converging to an operator* $K \in \mathcal{B}(\mathcal{H})$. *If the double limit*

$$\lim_{\lambda\to\partial_{\mathcal{H}}\Omega,\ n\to\infty} \widetilde{U^{-1}K_nU}(\lambda) := \ell$$

exists for any unitary operator $U \in \mathcal{B}(\mathcal{H})$, *then* K *is a compact operator and* $\ell = 0$ *holds.*

Proof. Since $K_n \to K$ weakly as $n \to \infty$, we have $\lim_{n\to\infty} \langle K_nf, g\rangle = \langle Kf, g\rangle$ for any $f, g \in \mathcal{H}$. In particular, $\lim_{n\to\infty} \left\langle K_n\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle = \left\langle K\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle$, which means that

$$\lim_{n\to\infty} \widetilde{K_n}(\lambda) = \widetilde{K}(\lambda), \quad \lambda \in \Omega.$$

On the other hand, we obtain that

$$\begin{aligned}\widetilde{U^{-1}K_nU}(\lambda) &= \left\langle K_nU\widehat{k}_{\mathcal{H},\lambda}, U\widehat{k}_{\mathcal{H},\lambda}\right\rangle \to \left\langle KU\widehat{k}_{\mathcal{H},\lambda}, U\widehat{k}_{\mathcal{H},\lambda}\right\rangle \\ &= \left\langle U^{-1}KU\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda}\right\rangle = \widetilde{U^{-1}KU}(\lambda)\end{aligned}$$

for every unitary operator $U \in \mathcal{B}(\mathcal{H})$. Thus

$$\widetilde{U^{-1}K_nU}(\lambda) \to \widetilde{U^{-1}KU}(\lambda) \text{ as } n \to \infty$$

for all $\lambda \in \Omega$. By considering that K_n, $n \geq 1$, are compact operators and $\partial_{\mathcal{H}}\Omega \neq \emptyset$, it follows from Theorem 2 that

$$\lim_{\lambda\to\partial_{\mathcal{H}}\Omega} \widetilde{U^{-1}K_nU}(\lambda) = 0.$$

Since by condition of the theorem the double limit

$$\lim_{\lambda\to\partial_{\mathcal{H}}\Omega,\ n\to\infty} \widetilde{U^{-1}K_nU} = \ell$$

exists and

$$\lim_{n\to\infty} \widetilde{U^{-1}K_nU}(\lambda) = \widetilde{U^{-1}KU}(\lambda),$$

we obtain

$$\lim_{n\to\infty}\lim_{\lambda\to\partial_{\mathcal{H}}\Omega}\widetilde{U^{-1}K_nU}(\lambda)=\lim_{\lambda\to\partial\Omega}\lim_{n\to\infty}\widetilde{U^{-1}K_nU}(\lambda)=\ell,$$

which means that $\lim_{\lambda\to\partial_{\mathcal{H}}\Omega}\widetilde{U^{-1}KU}(\lambda)=\ell=0$, because it is obvious that

$$\lim_{n\to\infty}\lim_{\lambda\to\partial_{\mathcal{H}}\Omega}\widetilde{U^{-1}K_nU}(\lambda)=0.$$

Now by applying Theorem 2, we obtain that K is compact, as desired. □

4 Invariant subspaces of compact operators

The famous theorems of J. von Neumann (see in [2]) and Lomonosov [7] prove in particular the existence of nontrivial invariant subspaces of compact operators on the Hilbert and Banach spaces. Here we describe in terms of reproducing kernels and Berezin symbols invariant subspaces of compact operators. The following result belongs to Karaev [5, Proposition 3.5].

Proposition 1. *Let $\varphi\in L^\infty(\partial\mathbb{D})$ and T_φ be a Toeplitz operator defined on the Hardy-Hilbert space $H^2=H^2(\mathbb{D})$ by $T_\varphi f=P_+\varphi f$, where $P_+:L^2(\partial\mathbb{D})\to H^2$ is the Riesz projector. Then for every (closed) subspace $E\subset H^2(\mathbb{D})$*

$$\left|\left(T_\varphi k^E_{H^2,\lambda}\right)(\lambda)-\left(\widetilde{\varphi}k^E_{H^2,\lambda}\right)(\lambda)\right|=O\left(\frac{1}{1-|\lambda|^2}\right)\tag{1}$$

as $\lambda\to\partial\mathbb{D}$ radially; here $\widetilde{\varphi}$ is the harmonic extension of φ onto the unit disc $\mathbb{D}$, $k^E_{H^2,\lambda}:=P_Ek_{H^2,\lambda}$ is the reproducing kernel of the subspace E and $k_{H^2,\lambda}(z)=(1-\overline{\lambda}z)^{-1}$ stands for the reproducing kernel of H^2.

We have from this result in particular that if $E\subset H^2(\mathbb{D})$ is a nontrivial (i.e., $\{0\}\neq E\neq H^2$) closed T_φ-invariant subspace, (i.e., $T_\varphi E\subset E$) then the relation (1) is satisfied. The proof of Proposition 1 shows that the latter corollary is also true for the invariant subspaces of any compact operators K acting on H^2. In this section, we generalize this result for the compact operators acting in any RKHS with $\partial_H\Omega\neq\emptyset$ (see Section 3).

Theorem 4. *Let $\mathcal{H}=\mathcal{H}(\Omega)$ be a RKHS on Ω such that $\partial_{\mathcal{H}}\Omega\neq\emptyset$, and let $K\in\mathcal{B}(\mathcal{H})$ be a compact operator. If $E\subset\mathcal{H}$ be a subspace and $KE\subset E$, then*

$$\lim_{\lambda\to\xi\in\partial_H\Omega}\frac{\left|\left(Kk^E_{H,\lambda}\right)(\lambda)-\left(\widetilde{K}k^E_H\right)(\lambda)\right|}{\|k_{H,\lambda}\|^2}=0,\tag{2}$$

where $\widetilde{K}$ is the Berezin symbol of the operator K.

Proof. The proof is similar to the one of Proposition 1 from [5]; however, for completeness reasons, we provide it. For this purpose, let $(e_j(z))_{j\geq 1}$ be an orthonormal basis in E. Then, clearly $P_E f = \sum_{j\geq 1} \langle f, e_j\rangle e_j$ for every $f \in \mathcal{H}$, where $P_E : \mathcal{H} \to E$ is an orthogonal projection. By setting

$$g := K(P_E k_{\mathcal{H},\lambda}) - \widetilde{K}(\lambda)(P_E k_{\mathcal{H},\lambda}),$$

we have

$$g = (K - \widetilde{K}(\lambda) I_{\mathcal{H}}) \sum_{j\geq 1} \langle k_{\mathcal{H},\lambda}, e_j\rangle e_j,$$

$$\begin{aligned} g(\lambda) = \langle g, k_{\mathcal{H},\lambda}\rangle &= \left\langle (K - \widetilde{K}(\lambda) I_{\mathcal{H}}) \sum_{j\geq 1} \langle k_{\mathcal{H},\lambda}, e_j\rangle e_j, k_{\mathcal{H},\lambda} \right\rangle \\ &= \left\langle \sum_{j\geq 1} \langle k_{\mathcal{H},\lambda}, e_j\rangle e_j, \left(K^* - \overline{\widetilde{K}}(\lambda) I_{\mathcal{H}}\right) k_{\mathcal{H},\lambda} \right\rangle, \end{aligned}$$

and hence

$$\begin{aligned} |g(\lambda)| &\leq \left\| \sum_{j\geq 1} \langle k_{\mathcal{H},\lambda}, e_j\rangle \right\| \left\| \left(K^* - \overline{\widetilde{K}}(\lambda) I_{\mathcal{H}}\right) k_{\mathcal{H},\lambda} \right\| \\ &\leq \|k_{\mathcal{H},\lambda}\|_{\mathcal{H}} \left\| \left(K^* - \overline{\widetilde{K}}(\lambda) I_{\mathcal{H}}\right) k_{\mathcal{H},\lambda} \right\| \\ &= \|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}^2 \left\| \left(K^* - \overline{\widetilde{K}}(\lambda) I_{\mathcal{H}}\right) \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}} \right\| \\ &= \|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}^2 \left\| \left(K^* - \overline{\widetilde{K}}(\lambda) I_{\mathcal{H}}\right) \widehat{k}_{\mathcal{H},\lambda} \right\|. \end{aligned}$$

Thus

$$|g(\lambda)| \leq \|k_{\mathcal{H},\lambda}\|_{\mathcal{H}}^2 \left\| \left(K^* - \overline{\widetilde{K}}(\lambda) I_{\mathcal{H}}\right) \widehat{k}_{\mathcal{H},\lambda} \right\|. \tag{3}$$

Now, by considering that

1. $\partial_{\mathcal{H}}\Omega \neq \emptyset$;

2. K is compact (and hence K^* is also compact);

3. $\widetilde{K}(\lambda) \leq \left\| K\widehat{k}_{\mathcal{H},\lambda} \right\|_{\mathcal{H}}$;

4. $\widehat{k}_{\mathcal{H},\lambda} \to 0$ weakly as $\lambda \to \xi \in \partial_{\mathcal{H}}\Omega$ (see Theorem 2 in Section 3);

we have that $\left\| \left(K^* - \overline{\widetilde{K}}(\lambda) I_{\mathcal{H}}\right) \widehat{k}_{\mathcal{H},\lambda} \right\| \to 0$ as $\lambda \to \xi \in \partial_{\mathcal{H}}\Omega$. According (3), this implies (2). The proof is completed. □

Corollary 1. *For $\varphi \in L^{\infty}(\partial\mathbb{D})$, let T_{φ} be a Toeplitz operator on H^2 and $K : H^2 \to H^2$ be any compact operator. If $E \subset H^2$ and $(T_{\varphi} + K)\, E \subset E$, then*

$$\lim_{\lambda \to \partial\mathbb{D}} \left(1 - |\lambda|^2\right) \left| \left[(T_{\varphi} + K)\, P_E \left(\frac{1}{1 - \bar{\lambda} z} \right) \right] (\lambda) - \left(\widetilde{\varphi}(\lambda) + \widetilde{K}(\lambda) \right) \left(P_E \frac{1}{1 - \bar{\lambda} z} \right) \right| = 0.$$

This corollary is immediate from Proposition 1 and Theorem 4.

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References

[1] Aronzajn N., Theory of reproducing kernels, Trans. Amer. Math. Soc., 68(1950), 337-404.

[2] Aronzajn N., and Smith K.T., Invariant subspaces of completely continuous operators, Ann. of Math., 60(1954), 345-350.

[3] Chalendar I., Fricain E., Gürdal M. and Karaev M. T., Compactness and Berezin symbols, Acta Sci. Math. (Szeged), 78(2012), 315-329.

[4] Gohberg I. Ts. and Krein M.G., Introduction to Theory of Linear Nonselfadjoint Operators, Translations of Mathematical Monographs 18, American Mathematical Society, Providence, RI, 1969.

[5] Karaev M. T., Use of reproducing kernels and Berezin symbols technique in some questions of operator theory, Forum Math., 24(2012), 553–564.

[6] Karaev M. T., Reproducing kernels and Berezin symbols techniques in various questions of operator theory, Complex Anal. Oper.Theory, 7(2013), 983-1018.

[7] Lomonosov V. I., Invariant subspaces of the family of operators that commute with a completely continous operator, Funct. Anal. Appl., 7(1973), 213-214.

[8] Nordgren E. and Rosenthal P., Boundary values of Berezin symbols, Oper. Theory: Advances and Applications, 73(1994), 362-368.

[9] Pehlivan S., and Karaev M.T., Some results related with statistical convergence and Berezin symbols, J. Math. Anal. Appl., 299(2004), 333-340.

[10] Saitoh S., Theory of Reproducing Kernels and its Applications, Pitman Research Notes in Mathematics 189, Longman Scientific Technical, 1988.

[11] Saitoh S., Integral transforms, Reproducing Kernels and their Applications, Pitman Res. Notes in Math. Series 369, Addison Wesley Longman Ltd, UK, 1997.

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