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RELATION BETWEEN GROUPS WITH BASIS PROPERTY AND GROUPS WITH EXCHANGE PROPERTY

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Abstract

A group G is called a group with basis property if there exists a basis (minimal generating set) for every subgroup H of G and every two bases are equivalent. A group G is called a group with exchange property, if $x \notin \langle X \rangle \wedge x \in \langle X \cup \{y\} \rangle$, then $y \in \langle X \cup \{x\} \rangle$, for all $x, y \in G$ and for every subset $X \subseteq G$.

In this research, we proved the following: Every polycyclic group satisfies the basis property. Every element in a group with the exchange property has a prime order. Every p-group satisfies the exchange property if and only if it is an elementary abelian p-group. Finally, we found necessary and sufficient condition for every group to satisfy the exchange property, based on a group with the basis property.

1 Introduction

A generating set X is said to be minimal if it has no proper subset which forms a generating set. The subset X of a group G is called independent, if for all $x \in X,\ x \notin \langle X \setminus \{x\} \rangle$. Independent set X is called a basis subgroup $\langle X \rangle$. In 1978 Jones [5] introduced an initial study of semigroups with the basis property. Jones [5] states that if G is an inverse semigroup and $U \leq V \leq G$ then a U-basis for V is a subset X of V which is minimal such that $\langle U \cup X \rangle = G$.

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So a minimal generating set for V is a $\emptyset$-basis. A basis property of universal algebra A means that every two minimal (with respect to inclusion) generating set (basis) of an arbitrary subalgebra of A have the same cardinality [1].

2 Basis property

Definition 2.1 A group G is called a group with basis property if there exists a basis minimal (irreducible) generating sets (with respect to inclusion) for every subgroup H of G and every two bases are equivalent (i.e. they have the same cardinality) [1].

Notice that finitely generated vector spaces have the property that all minimal generating sets have the same cardinality. Jones [5] introduce another concept which is state for inverse semigroup.

Definition 2.2 An inverse semigroup S has the strong basis property if for any inverse subsemigroup V of S and inverse subsemigroup U of V any two U-bases for V have the same cardinality.

Let $(\mathbb{Z}, +)$ be an additive abelian group, then we can write $\mathbb{Z} = \langle 1 \rangle = \langle 2, 3 \rangle$ even though $2 \notin \langle 3 \rangle$ and $3 \notin \langle 2 \rangle$. Thus $\mathbb{Z}$ does not have the basis property. Hence free groups do not have the basis property. The first results on the basis property of groups was in [6]. The author proved that a group with basis property is periodic, all elements of such a group have prime power order, and solvable. Therefore by [1] every finite p-group has a basis property, and the homomorphic image of every finite group with basis property is again a group with basis property, but in case of infinite group we have the following:

Remark 2.3 Let $G = \sum_{i=1}^{\infty} \mathbb{Z}_{p^i}$ be a direct sum of a cyclic p-group P, then one of homomorphic image is a quasicyclic group $K = \mathbb{Z}_{p^\infty}$, which is not a group with basis property, but the group G is a group with basis property.

Lemma 2.4 Let G be a group in which every element has prime power order, let $x \in G$ such that $|x| = p^c$ and $y \in G$ such that $|y| = q^{\ b}$, $p \neq q$ are primes. Then $xy \neq yx$.

Proof. Suppose that $xy = yx$, then xy is an element of order $p^c q^b$, hence xy has a composite order in G. This is contradiction with basis property[1], so $xy \neq yx$.

□

Proposition 2.5 Let G be a finite nilpotent group. Then G is a group with basis property if and only if G is a primary group.

Proof. Suppose that G is a finite nilpotent group with basis property. From [11] every finite nilpotent group is decomposable in a direct product of Sylow subgroups. Then

$$G = G_1 \times G_2 \times \cdots \times G_m,$$

such that G_i is a p_i-group for some primes p_i, $p_i \neq p_j$ if $i \neq j$. If $m > 1$, then in G there exists two commute elements with a prime power order. Hence we have a contradiction with lemma(2-4). Thus G is a primary group.

Conversely, if G is a primary group, then G is a group with basis property [5]. □

A classification of group with the basis property was announced by Dickson and Jones in [5], but as far as we can see this has yet to be published. However a classification of finite groups with the basis property was given by Al Khalaf [1] exploiting Higman's result, this classification requires a technical condition on the p-group and he proved the following theorem:

Theorem 2.6 [1]. Let a finite group G be a semidirect product of a p-group $P = Fit(G)$ (Fitting subgroup) of G by a cyclic $q-$ group $\langle y \rangle$, of order q^b, where $p \neq q$ (p and q are primes), $b \in \mathbb{N}$.Then the group G has basis property if and only if for any element $y \in \langle y \rangle$, $y \neq e$ and for any invariant subgroup $H$of P the automorphism φ_u must define an isotopic representation on every quotient Frattini subgroup of H.

In [4], the author used some common results from both group and module theory using Maschke, Clifford and Krull-Schmidt, to classify the group with basis property.

Finally Jones [7] studied basis property from the point of view exchange properties.

Theorem 2.7 [3] Let G be a semidirect product of abelian p-group P by a cyclic q-group $\langle y \rangle$, of order q^b, where $p \neq q$ (p and q are primes), $b \in \mathbb{N}$, which is defined automorphism φ of orderq^b and P has an exponent p^c , $c \in \mathbb{N}$. Then the group G has basis property if and only if there exists a polynomial $g(x) \in \mathbb{Z}[x]$ such that satisfy the following conditions:

1. The polynomial $f(x) = \overline{\theta}(g(x))$ is irreducible over the field $GF(p)$, $f(x) \mid x^{q^b} - 1$ and $f(x) \nmid x^{q^{b-1}} - 1$.
2. $g^m(\varphi) = 0$.

In this research we study special group with the basis property. The concept of exchange property and continued results as shown in [7] and[8].

Theorem 2.8 Let G be a finite polycyclic group such that G has a presentation [9]:

$$G = \left\langle x, y \ : x^{p^c} = y^{q^b} = 1 \ , \ y^{-1}xy = x^r \right\rangle , \qquad (2-1)$$

such that $p \neq q$ (p and q are primes) $b, c, r \in \mathbb{Z}^+, (p \ , \ \ r-1) = 1$ and

$$r^{q^b} \ \equiv 1 \ (\ mod \ p^c), r \not\equiv 1 \ (\ mod \ p \ ) \ , 0 \leq r \leq p^c. \qquad (2-2)$$

Then G is a group with the basis property if and only if it satisfies the following conditions:

$$p \ \equiv 1 \ \left(\ mod \ q^b \right) , \qquad (2-3)$$

$$r^{q^{b-1}} \not\equiv 1 \ \ (\ mod \ p \ ) \ . \qquad (2-4)$$

Proof. Suppose that G is a group with the basis property. From $(2-1)$ we have that G is a semidirect product of cyclic p-group $\langle x \rangle$, $|\langle x \rangle| = p^c$ by a cyclic q-group $|\langle y \rangle| = q^b$, where $p \neq q$ (p and q are primes) $b, c \in \mathbb{Z}^+$. Then from [1] G is a Frobenius group with kernel $\langle x \rangle$ and complement $\langle y \rangle$. Thus by [3] we see that $p \ \equiv 1 \ \left(\ mod \ q^b \right)$. Thus $(2-3)$ holds.

Assume that

$$r^{q^{b-1}} \ \ \equiv 1 \ \ (\ mod \ p \ ) \ . \qquad (2-5)$$

Then $r^{q^{b-1}} = 1 + mp$ for some $m \in \mathbb{Z}^+$.Considering the non trivial elements $x^{p^{c-1}}$, $y^{q^{b-1}}$ and using $(2-1)$ and $(2-5)$ then we have:

$$y^{-q^{b-1}} \ x^{p^{c-1}} y^{q^{b-1}} = \ \left(y^{-q^{b-1}} \ \ xy^{q^{b-1}} \right)^{p^{c-1}} = \left(y^{-q^{b-1}-1} \left(y^{-1} \ xy \right) y^{q^{b-1}-1} \right)^{p^{c-1}}$$

$$= \left(y^{-q^{b-1}-1} x^r y^{q^{b-1}-1} \right)^{p^{c-1}} = \cdots = \left(x^{r^{q^{b-1}}} \right)^{p^{c-1}} = x^{p^{c-1}(1+mp)} =$$

$$x^{p^{c-1}} (\ x^{p^c})^m = x^{p^{c-1}}.$$

Hence the p-element $x^{p^{c-1}}$ commutes with the q- element in G, so we have a contradiction with lemma (2-4). Thus $(2-4)$ holds.

Conversely, let G be a polycyclic group satisfying conditions $(2-3)$, and $(2-4)$.Then from [9] we see that G is an extension of cyclic p-group $\langle x \rangle$ of order $p^{\ c}$ by cyclic q-group $\langle y \rangle$ of order q^b, $p \neq q$ (p and q are primes) $b, c \in \mathbb{Z}^+$. Thus $(|\langle x \rangle| \, , |\langle y \rangle| \ ) = 1$ and $|G| = |\langle x \rangle| \, |\langle y \rangle|$, then $\langle x \rangle \cap \langle y \rangle = \{1\}$ and $G = \langle x \rangle \, \langle y \rangle$, so $G = \langle x \rangle \rtimes \langle y \rangle$. Since $\langle x \rangle \trianglelefteq G$ and $\langle x \rangle$ is an abelian p-group, then by using theorem (2-7) we prove that G is a group with the basis property.

Now consider the polynomial $g(x) = x - r$ over the ring $\mathbb{Z}$. Denote that

$f(x) = \overline{\theta}(g(x))$. Then the polynomial $f(x)$ is an irreducible over the field $GF(p)$ and has $\overline{r}$ zeros.Thus by $(2-2)$, and $(2-4)$ we have $\overline{r}^{q^b} = 1$, $\overline{r}^{q^{b-1}} \neq 1$, hence by Bezout theorem the polynomial $f(x)$ is divides $x^{q^b}-1$ and not divides $x^{q^{b-1}}-1$, i.e. the condition 1) in theorem(2-7) holds for$g(x)$. Now consider the automorphism φ, which defines a semidirect product $\langle x \rangle \rtimes \langle y \rangle$ and induced by y element, i.e.

$$\varphi : a \to y^{-1}ay, \quad \forall\, a \in \langle x \rangle .$$

From $(2-1)$ we get

$$\varphi(a) = a^r, \quad \forall\, a \in \langle x \rangle .$$

Using additive form in $\langle x \rangle$, then we have $g(\varphi) = 0$. Thus the codition 2) of theorem(2-7) for $g(x)$ holds too. Hence G is a group with the basis property. □

3 Exchange property

The fundamental property of generating operator φ of subspace of the vector space V over the field F that this operator satisfies exchange property.

Definition 3.1 Let V be a vector space, then $\forall\, x, y \in V$ and for every subset $X \subseteq V$ if $x \notin \varphi(X)$ and if $x \in \varphi(X \cup \{y\})$, then $y \in \varphi(X \cup \{x\})$.

Theorem 3.2 Let G be a group with the exchange property, i.e. $\forall\, x, y \in G$ and for every subset $X \subseteq G$,

$$\text{if } x \notin \langle X \rangle \wedge x \in \langle X \cup \{y\} \rangle \text{ , then } y \in \langle X \cup \{x\} \rangle . \qquad (3-1)$$

Then the order of every element $a \in G$, $a \neq 1$ is a prime.

Proof. First, we prove that every cyclic subgroup of G is simple, i.e. every cyclic subgroup does not contain non trivial normal subgroup.

Suppose that $\{1\} \leq \langle x \rangle \leq \langle y \rangle$ for $x, y \in G$. Then $x \notin \{1\}$ and $x \in \langle \{1\} \cup \{y\} \rangle$ such that substituting $X = \{1\}$ in $(3-1)$ we find $y \in \langle \{1\} \cup \{x\} \rangle = \langle x \rangle$ and we get a contradiction with our assumption. Thus $O(x) \in \{p,\ q\ \}$, $\forall\, x \in G \backslash \{1\}$.

Theorem 3.3 Let G be a p-group such that p is a prime. Then G is a group with the exchange property if and only if G is elementary abelian p-group.

Proof. Suppose that G is a p-group with the exchange property. Then by theorem (3-2)

$$x^{\,p} = 1\ ,\ \forall\, x \in G, \qquad (3-2)$$

hence $G^p = \{1\}$ and by [10] $\Phi(G) = G^pG'$. Since G is a p-group, then

$$\Phi\,(G) = G',\ \ \Phi^2\,(G) = G'', \ldots.$$

If $G' = \{1\}$, then G is an elementary abelian group.

Suppose that $G' \neq \{1\}$. Then there exist elements $a, b, c \in G$ such that

$$[a, b] = a^{-1}b^{-1}ab = c \neq 1. \qquad (3-3)$$

Now assume that $c \in \langle a\rangle$, then $a \in \langle c\rangle$. Let consider the subgroup, which is generated by two elements a, b, i.e. $\langle a, b\,\rangle$. If $\langle a, b\,\rangle$ is a cyclic group, then it is commutative and we have a contradiction with $(3-3)$, then $a \notin \langle b\,\rangle$ and $b \notin \langle a\,\rangle$. Hence the set $\{a, b\,\}$ forms a basis of group $\langle a, b\,\rangle$. Since $\langle a\,\rangle = \langle c\,\rangle$, so $\langle a, b\,\rangle = \langle c, b\,\rangle$ and by the basis property of G [6]. Thus we have that the set $\{\ c, b\ \}$ forms a basis of G and this is a contradiction with properties of the Frattini subgroup, i.e.$c \in \Phi\,(G)$.

Hence $c \notin \langle \text{a}\rangle$ and $c \in \langle a, b\,\rangle$, and by the exchange property we have $b \in \langle a, c\,\rangle$. But then $\langle a, b\,\rangle = \langle a, c\,\rangle$. So by the basis property for G and since $a \notin \langle \text{b}\rangle$, $b \notin \langle \text{a}\rangle$ we conclude that the set $\{a, c\}$ forms a basis for G. Hence this is a contradiction with properties of the the Frattini subgroup $\Phi\,(G)$, i.e. $c \in \Phi\,(G)$. Thus $[a, b] = 1$ and the group G is an elementary abelian p-group.

Conversely, suppose that a group G is an elementary abelian p-group, then we consider G as an additive group of a vector space over the field $GF(p)$.

Hence the exchange property is satisfied for a group G.

4 Intersection between the basis property and the exchange property

Example 4.1 Let S be the semilattice $\{\ a, b, 0\ \}$, where a, b are incomparable and $ab = 0$. Then S has unique basis, so S has basis property. But $0 \in \langle\langle a\rangle \cup \{b\}\rangle$ and $0 \notin \langle\langle a\rangle\rangle$, $b \notin \langle\langle a\rangle \cup \{0\}\rangle$. Hence S does not satisfy the exchange property.

Example 4.2 Let $G = \langle a\rangle$ be a cyclic group such that $|G\ | = p^2$, p is a prime. Then G is a group with the basis property, because it is a p-group, but it does not satisfy the exchange property.

Theorem 4.3 Let G be a finite group. Then G is a group with the exchange property if and only if one of the following conditions hold:

1. G is an elementary abelian p-group, p is a prime.
2. G is a semidirect product of an elementary abelian p-group P by a cyclic q-group $\langle y\rangle$, of order q, where $p \neq q$ (p and q are primes).Therefore G must satisfy the following relations:

$$p \equiv 1 \ (\ mod\ q), \quad y^{-1}ay = a^r \ , \ r \in \mathbb{Z}^+,$$
$$r \not\equiv 1 \ (\ mod\ p\ ), \ r^q \equiv 1 \ (\ mod\ p\ ).$$

Proof. Suppose that G is a group with the exchange property. Then we consider two cases:

Firstly, if G is a primary group(p-group), p is a prime, then by theorem(3-3) G is an elementary abelian p-group for a prime p.

Secondly, if G is not primary group, then from the basis property in theorem(2-6), we see that G is a semidirect product (i.e. $G = P \rtimes \langle y\rangle$) of p-group P by a cyclic q-group $\langle y\rangle$, where $p \neq q$ (p and q are primes). Since P is a group with the exchange property, then by theorem(3-3) P is an elementary abelian $p-$ group. Therefore by theorm(3-1) the group $\langle y\rangle$ has order q, q is a prime.

Suppose that $|P| = p^s$, $s \in \mathbb{Z}^+$. Since the element y is regular operator on P, i.e. the operator φ inducing by element y is a regular, then

$$p^s \equiv 1 \ (\ mod\ q) \ .$$

Assume that $a \in P$, $a \neq 1$. Consider the element $b = y^{-1}ay$, since the operator φ induced by element y is regular, then $b \neq a$. Assume that $b \in \langle a\rangle$, hence $b = a^r$, $r \not\equiv 1 \ (\ mod\ p\ )$. From $y^q = 1$ we have $a^{r^q} = 1$, i.e. $r^q \equiv 1 \ (\ mod\ p\ )$.

Now let $b \notin \langle a\rangle$, so by the exchange property if $b \in \langle y, a\rangle$, then $y \in \langle a, b\rangle \leq P$. We get a contradiction with $y \notin P$. Thus the automorphism $\varphi_y : P \to P$ is regular and act on a group $\langle a\rangle$ of order p , hence $p \equiv 1 \ (\ mod\ q)$ and $p > q$. Since G is a group with the basis property, then by theorem(2-6) the representation $y \to \varphi_y$ is an isotopic with dimension 1, i.e. the matrix A of linear operator φ_y in some basis of vector space P which contains s elements has the following form:

$$A = \begin{pmatrix} \overline{r} & 0\ldots & 0 \\ 0 & \overline{r}\ldots & 0 \\ 0 & 0 & \overline{r} \end{pmatrix},$$

such that $\overline{r}$ is an image of the element r under the conical homomorphism $\overline{\theta}:\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, then

$r \not\equiv 1 \ (\ mod\ p\ )$, and

$$r^q \equiv 1 \ (\ mod\ p\ ).$$

Conversely, If G is an elementary abelian p-group for a prime p, then G is a group with basis property. Using theorem(3-3), then it remains to prove that if P is an elementary abelian, and $\langle y \rangle$ has order q, where $p \neq q$ (p and q are primes), and if the following conditions hold

$$y^{-1}xy = x^r\ ,\ \forall x \in P,$$

$$p\ \equiv 1 \ (\ mod\ q),$$

$$r \not\equiv 1 \ (\ mod\ p\ ),$$

$$r^q \equiv 1 \ (\ mod\ p\ ). \qquad (4-1)$$

Then G is a group with the exchange property. Suppose that the set $X \subseteq G$ and $a, b \in G$ such that $a \notin \langle X \rangle$ and $a \in \langle X \cup \{b\}\ \rangle$. Now we prove that $b \in \langle X \cup \{a\}\ \rangle$.

Let $G_1 = \langle X \cup \{b\}\ \rangle$ and we study the following cases:

If $\langle X \cup \{b\}\ \rangle\ \leq\ P$, then $G_1\ \leq\ \ P$, G_1 satisfies the exchange property, because it is an elementary abelian p-group (by theorem(3-3)), hence $b \in \langle X \cup \{a\}\ \rangle$.

If $\langle X \cup \{b\}\ \rangle \nleq P$, then suppose that the set $X \cup \{b\}$ contains element of order q. Now if$X$ contains elements of order q, and since G is a semidirect product of p-group by cyclic $\langle y\ \rangle$. Then we can prove that the set X contains only one element of order q, (because if there exist two elements as $y^{s_1}a_1$, $y^{s_2}a_2$ in X of order q, then for some $w \in \mathbb{Z}$ there exists $c \in P$ such that

$$y^{s_2}a_2 = (y^{s_1}a_1)^w\ c.$$

Then $\langle y^{s_1}a_1,\ \ c\rangle = \langle y^{s_1}a_1,\ \ y^{s_2}a_2\rangle$, hence we consider element $y^{s_2}a_2$ as $c \in P$). Now suppose that the set $X = \{x_1, x_2, \ldots, x_n\}$ such that $x_2, \ldots, x_n \in P$, $x_1 \notin P$. Then the Fitting subgroup $F(\langle X\ \rangle)$ of group $\langle X\ \rangle$ is generated by the set $\{x_2, \ldots, x_n\}$ and the image of this set under the automorphism $\varphi^m_{x_1}$, $m \in \mathbb{Z}$.

Since the group P is an abelian group, then the Fitting subgroup $F(\langle X\ \rangle)$ is generated by the set $\{x_2, \ldots, x_n\}$ and the image of this set under the automorphism φ^m_y and by $(4-1)$ this is the power of the same elements $x_2, \ldots, x_n$. In another words, the group $F(\langle X\ \rangle)$is generated by $x_2, \ldots, x_n$ if these elements are exists. So by our assumption$a \in \langle X \cup \{b\}\ \rangle$. Then there exists a word $u(x_1, x_2, \ldots, x_n)$ such that $a = u(x_1, x_2, \ldots, x_n, b)$ and by $(4-1)$ we have

$$a = v(x_1, x_2, \ldots, x_n)b^w, \qquad (4-2)$$

such that $v(x_1, x_2, \ldots, x_n)$ is a word. If $b^w = e$, then by $(4-2)$ we have

$$a = v(x_1, x_2, \ldots, x_n) \in \langle X \ \rangle \ .$$

Thus we get a contradiction with our assumption for a, so we assume that $b^w \neq e$. Since a group P is an elementary abelian p-group, then $\langle b^w \rangle = \langle b \rangle$, so by (4-2) we have

$$b \in \langle b^w \rangle = \left\langle v(x_1, x_2, \ldots, x_n)^{-1} a \right\rangle \subseteq \langle X \cup \{a\} \rangle \ .$$

Finally, let $X \subseteq P$. Since $X \cup \{b\} \nsubseteq P$, then b is element of order q. Suppose that $G_1 = \langle X \cup \{b\} \ \rangle$ is a semidirect product of a group $\langle X \ \rangle$ by $\langle b \ \rangle$. Then from $a \in \langle X \cup \{b\} \ \rangle$ we have the following for $w \in \mathbb{Z}$ and $c \in \langle X \ \rangle$

$$a = b^w c. \qquad (4-3)$$

If an element a is a q-element, then $b^w \neq e$ and since $\langle b \rangle = \langle b^w \rangle$ we get

$$b \in \langle b^w \rangle = \left\langle c^{-1} a \right\rangle \subseteq \langle X \cup \{a\} \rangle .$$

If a is $p-$element, then by $(4-3)$ we have $b^w = e$ and $a = c \in \langle X \rangle$ which is a contradiction with $a \notin \langle X \rangle$. Thus we study all cases. Hence G is a group with change property.

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