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Closed graphs are proper interval graphs

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Abstract

Let G be a connected simple graph. We prove that G is a closed graph if and only if G is a proper interval graph. As a consequence we obtain that there exist linear-time algorithms for closed graph recognition.

Introduction

In this note a graph G means a connected simple graph without isolated vertices, that is, G is connected without loops and multiple edges. Let $V(G) = [n] = \{1, \ldots, n\}$ be the set of vertices and $E(G)$ the edge set of G.

Let $S = K[x_1, \cdots, x_n, y_1, \cdots, y_n]$ be the polynomial ring in $2n$ variables with coefficients in a field K. For $i < j$, set $f_{ij} = x_i y_j - x_j y_i$. The ideal J_G of S generated by the binomials $f_{ij} = x_i y_j - x_j y_i$ such that $i < j$ and $\{i, j\}$ is an edge of G, is called *the binomial edge ideal* of G. Such class of ideals is a generalization of the ideal of 2-minors of a $2n$-matrix of indeterminates. In fact, the ideal of 2-minors of a $2n$-matrix may be considered as the binomial edge ideal of a complete graph on $[n]$. The relevance of this class of ideals for algebraic statistics is underlined in [14]. Indeed these ideals arise naturally in the study of conditional independence statements [5]. If $\prec$ is a monomial order on S, then a graph G on the vertex set $[n]$ is *closed with respect to the given labelling of the vertices* if the generators f_{ij} of J_G form a quadratic Gröbner basis [14, 4].

A combinatorial description of this fact is the following. A graph G is *closed with respect to the given labelling of the vertices* if the following condition

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is satisfied: for all edges $\{i, j\}$ and $\{k, \ell\}$ with $i < j$ and $k < \ell$, one has $\{j, \ell\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = \ell$.

In particular, G is *closed* if there exists a labelling for which it is closed.

In the last years different authors [14, 16, 4, 19] concentrated their attention on the class of closed graphs. The most recent characterization of this class of graphs is given in [3], where it is proved that a connected graph has a closed labeling if and only if it is chordal, $K_{1,3}$-free, and has a property called *narrow*, which holds when every vertex is distance at most one from all longest shortest paths of the graph.

In [4] we have conjectured that by a suitable ordering on the vertices it is possible to test the closedness of a graph in linear time. In this note we are able to prove the conjecture.

In the research of a linear-time algorithm for closed graph recognition we have observed that the class of closed graphs and the class of *proper interval graphs* are the same.

Proper interval graphs are the intersection graphs of intervals of the real line where no interval properly contains another and have been extensively studied since their inception [10, 12]. There are several representations and many characterizations of them [8, 13, 18] and some of them through vertex orderings. Such class of graphs has many applications, such as physical mapping of DNA and genome reconstruction [25, 9].

During the last decade, many linear-time recognition algorithms for proper interval graphs have been developed [2, 20, 17, 22] and most of them are based on special breadth-first search (BFS) strategies.

The first linear-time algorithm for interval graph recognition appeared in 1976 [1]. This algorithm uses a *lexicographic breadth first search* (lexBFS) to find in linear time the maximal cliques of the graphs and then employs special structure called PQ-trees to find an ordering of the maximal cliques that characterizes interval graphs. A lexBFS is a breadth first search procedure with the additional rule that vertices with earlier visited neighbors are preferred and its vantage is that it can be performed in $O(|V(G)| + |E(G)|)$ time [24].

The paper is organized as follows. Section 1 contains some preliminaries and notions that will be used in the paper. In Section 2, we prove our conjecture (Theorem 2.4): *Let G be a graph. G is a closed graph if and only if G is a proper interval graph.*

As a consequence we are able to state that by an ordering on the vertices obtained by a lexBFS research it is possible to test the closedness of a graph in linear-time.

1 Preliminaries

In this Section we recall some concepts and a notation on graphs and simplicial complexes that we will use in the article.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.

When we fix a given labelling on the vertices we say that G is a graph on $[n]$.

Let G be a graph with vertex set $[n]$. A subset C of $[n]$ is called a *clique* of G is for all i and j belonging to C with $i \neq j$ one has $\{i, j\} \in E(G)$.

Two graphs G and H are isomorphic if there exists a bijection between the vertex sets of G and H, namely $\phi : V(G) \to V(H)$, such that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$.

Set $V = \{x_1, \dots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that

(i) $\{x_i\} \in \Delta$ for all $x_i \in V$ and

(ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$.

An element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ. If Δ is a simplicial complex with facets $F_1, \dots, F_q$, we write $\Delta = \langle F_1, \dots, F_q \rangle$.

Definition 1.1. The *clique complex* $\Delta(G)$ of G is the simplicial complex whose faces are the cliques of G.

The clique complex plays an important role in the study of the class of *closed graphs* [14, 4].

Definition 1.2. A graph G is *closed with respect to the given labelling* if the following condition is satisfied:

for all edges $\{i, j\}$ and $\{k, \ell\}$ with $i < j$ and $k < \ell$ one has $\{j, \ell\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = \ell$.

In particular, G is *closed* if there exists a labelling for which it is closed.

Theorem 1.3. *Let G be a graph. The following conditions are equivalent:*

(1) *there exists a labelling $[n]$ of G such that G is closed on $[n]$;*

(2) *J_G has a quadratic Gröbner basis with respect to some term order $\prec$ on S;*

(3) *there exists a labelling of G such that all facets of $\Delta(G)$ are intervals $[a, b] \subseteq [n]$.*

Proof. (1) $\Leftrightarrow$ (2): see [4], Theorem 3.4.

(1) $\Leftrightarrow$ (3): see [16], Theorem 2.2. $\square$

2 The result

In this Section we prove that closed graphs are proper interval graphs and viceversa.

Definition 2.1. A graph G is an *interval graph* if to each vertex $v \in V(G)$ a closed interval $I_v = [\ell_v, r_v]$ of the real line can be associated, such that two distinct vertices $u, v \in V(G)$ are adjacent if and only if $I_u \cap I_v \neq \emptyset$.

The family $\{I_v\}_{v \in V(G)}$ is *an interval representation* of G.

Definition 2.2. A graph G is a *proper interval graph* if there is an interval representation of G in which no interval properly contains another.

If G is a graph, a *vertex ordering* σ for G is a permutation of $V(G)$. We write $u \prec_\sigma v$ if u appears before v in σ.

Ordering σ is called a *proper interval ordering* if for every triple u, v, w of vertices of G where $u \prec_\sigma v \prec_\sigma w$ and $\{u, w\} \in E(G)$, one has $\{u, v\}, \{v, w\} \in E(G)$. This condition is called the *umbrella property* [13].

The vertex orderings allow to state many characterizations of proper interval graphs. We quote the next result from [18, Theorem 2.1].

Theorem 2.3. *A graph G is a proper interval graph if and only if G has a proper interval ordering.*

Now we are in position to state and prove the result of the paper.

Theorem 2.4. *Let G be a graph. The following conditions are equivalent:*

(1) *G is a closed graph;*

(2) *G is a proper interval graph.*

Proof. Since a graph G is closed if and only if each connected component is closed we may assume that the graph G is connected.
(1)$\Rightarrow$ (2). Let G be a closed graph.

Claim 1. There exists a proper interval graph H such that G is isomorphic to H.

Since G is closed then there exists a labelling $[n]$ of G such that all facets of the clique complex $\Delta(G)$ are intervals $[a, b] \subseteq [n]$ (Theorem 1.3), that is

$$\Delta(G) = \langle [a_1, b_1], [a_2, b_2], \ldots, [a_r, b_r] \rangle, \tag{2.1}$$

with $1 = a_1 < a_2 < \ldots < a_r < n$, $1 < b_1 < b_2 < \ldots < b_r = n$ with $a_i < b_i$ and $a_{i+1} \leq b_i$, for $i \in [r]$.

Set $\varepsilon = \dfrac{1}{n}$. Define the following closed intervals of the real line:

$$I_k = [k, b(k) + k\varepsilon],$$

where

$$b(k) = \max\{b_i \,:\, k \in [a_i, b_i]\}, \qquad \text{for } k = 1, \dots, n. \tag{2.2}$$

Let H be the interval graph on the set $V(H) = \{I_1, \dots, I_n\}$ and let

$$\varphi : V(G) = [n] \to V(H)$$

be the map defined as follows:

$$\varphi(k) = I_k.$$

φ is an isomorphism of graphs.

In fact, let $\{k, \ell\} \in E(G)$ with $k < \ell$. We will show that $\{\varphi(k), \varphi(\ell)\} = \{I_k, I_\ell\} \in E(H)$, that is, $I_k \cap I_\ell \neq \emptyset$.

It is

$$I_k = [k, b(k) + k\varepsilon], \qquad I_\ell = [\ell, b(\ell) + \ell\varepsilon].$$

Suppose $I_k \cap I_\ell = \emptyset$. Then $b(k) + k\varepsilon < \ell$ and consequently $b(k) < \ell$. It follows that does not exist a clique containing the edge $\{k, \ell\}$. A contradiction.

Now, suppose that $\{I_k, I_\ell\} \in E(H)$, with $k < \ell$. We will prove that $\{k, \ell\} \in E(G)$.

Since $I_k \cap I_\ell \neq \emptyset$, then $b(k) + k\varepsilon \geq \ell$. By the meaning of ε and by the assumption $k < \ell$, it follows that $k\varepsilon < 1$ and so $b(k) \geq \ell$. Hence from (2.1) and (2.2), $\{k, \ell\} \in E(G)$.

Since G is closed and consequently a $K_{1,3}$-free graph [23], the isomorphism φ assures that H is a proper interval graph.

Hence G is up to isomorphism a proper interval graph and (2) follows.

(2)$\Rightarrow$ (1). Let G be a proper interval graph.

Claim 2. There exists a closed graph H such that G is isomorphic to H.

Let $\{I_v\}_{v \in V(G)}$ be an interval representation of G, with $|V(G)| = n$.

From Theorem 2.3, there exists a proper interval ordering σ of G. Let $\sigma = (I_1, \dots, I_n)$ be such vertex ordering. It is $I_j \prec_\sigma I_k$ if and only if $j < k$.

Let H be the graph with vertex set $V(H) = [n]$ and edge set $E(H) = \{\{i, j\} \,:\, \{I_i, I_j\} \in E(G)\}$.

We prove that H is a closed graph on $[n]$.

Let $\{i, j\}, \{k, \ell\} \in E(H)$ with $i < j$ and $k < \ell$.

Suppose $i = k$. Since $\{i, j\}, \{i, \ell\} \in E(H)$, then $\{I_i, I_j\}, \{I_i, I_\ell\} \in E(G)$.

If $i < j < \ell$, then $I_i \prec_\sigma I_j \prec_\sigma I_k$. Hence since σ satisfies the umbrella property and $\{I_i, I_\ell\} \in E(G)$, it follows that $\{I_i, I_j\}, \{I_j, I_\ell\} \in E(G)$. Thus $\{j, \ell\} \in E(H)$.

Repeating the same reasoning for $i < \ell < j$, it follows that $\{j, \ell\} \in E(H)$ again.

Similarly for $j = \ell$, one has $\{i, k\} \in E(H)$. Hence H is a closed graph.

It is easy to verify that the proper interval graph G is isomorphic to the closed graph H by the map $\psi : V(G) \to V(H) = [n]$, that sends every closed interval $I_j \in V(G)$ to the integer $j \in V(H)$.

Hence G is up to isomorphism a closed graph and (1) follows. □

Remark 2.5. For the implication (2) ⇒ (1), see also [19, Proposition 1.8]

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