DOI: 10.2478/auom-2014-0042
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 22(2),2014, 193–215

Numerical Decomposition of Affine Algebraic Varieties

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Abstract

An irreducible algebraic decomposition $\cup_{i=0}^{d} X_i = \cup_{i=0}^{d}(\cup_{j=1}^{d_i} X_{ij})$ of an affine algebraic variety X can be represented as a union of finite disjoint sets $\cup_{i=0}^{d} W_i = \cup_{i=0}^{d}(\cup_{j=1}^{d_i} W_{ij})$ called numerical irreducible decomposition (cf. [14],[15],[18],[19],[20],[22],[23],[24]). The W_i correspond to the pure i-dimensional components X_i, and the W_{ij} present the i-dimensional irreducible components X_{ij}. The numerical irreducible decomposition is implemented in Bertini (cf. [3]). The algorithms use homotopy continuation methods. We modify this concept using partially Gröbner bases, triangular sets, local dimension, and the so-called zero sum relation. We present in this paper the corresponding algorithms and their implementations in Singular (cf. [8]). We give some examples and timings, which show that the modified algorithms are more efficient if the number of variables is not too large. For a large number of variables Bertini is more efficient*.

1 Introduction

Given a system of n polynomials in $\mathbb{C}^N$,

$$f(x_1, ..., x_N) := \begin{pmatrix} f_1(x_1, ..., x_N) \\ \cdot \\ \cdot \\ f_n(x_1, ..., x_N) \end{pmatrix}.$$

Key Words: Witness point set, Homotopy function, Gröbner basis, Local dimension, Monodromy action, Zero Sum Relation.
2010 Mathematics Subject Classification: Primary 14Q15.
Received: September, 2012.
Revised: September, 2012.
Accepted: February, 2013.

Let $X = V(f)$ be the algebraic variety defined by the system above. X has a unique algebraic decomposition into d pure i-dimensional components X_i, $X = \cup_{i=0}^{d} X_i$. Where $X_i = \cup_{j_i}^{d_i} X_{ij}$ is empty or the union of d_i i-dimensional irreducible components.
The numerical irreducible decomposition (cf. [15],[18],[19],[20],[23]) is given as the union $W = \cup_{i=0}^{d} W_i = \cup_{i=0}^{d}(\cup_{j=1}^{d_i} W_{ij})$. The W_i are called i-witness point sets and are given as an intersection of the pure i-dimensional component X_i of X with a generic linear space L in $\mathbb{C}^N$ of dimension $N - i$. The finite sets W_{ij} are called the irreducible witness point sets representing the irreducible components X_{ij} of dimension i. The irreducible witness point sets have the following properties:

1. $W_{ij} \subset X_{ij}$,
2. $\sharp(W_{ij}) = deg(X_{ij})$,
3. $W_{ij} \cap W_{il} = \emptyset$ for $j \neq l$.

The computation of the numerical irreducible decomposition uses numerical homotopy continuation methods (cf. [25],[26]). This requires that the number n of polynomials of a given polynomial system is equal to the number N of variables.
The numerical irreducible decomposition proceeds in three steps:

The first step reduces the polynomial system to a system of N polynomials in N variables and computes a finite set $\widehat{W_i}$ called witness point super set for each non-empty pure i-dimensional component X_i. $\widehat{W_i}$ consists of points of X_i and J_i a set of points on components of larger dimension, the so-called junk point set (cf. [15],[18],[23]).

The second step removes the points of J_i from $\widehat{W_i}$ to obtain a subset W_i of the pure i-dimensional component X_i (cf. [23]).

The third step breakups W_i into irreducible witness point sets representing the i-dimensional irreducible components of X (cf. [14],[22]).

In [15],[18],[23] the cascade algorithm is used to compute the witness point super sets $\widehat{W_i}$. In the second section we modify this algorithm replacing the use of the homotopy function by Gröbner basis computations at certain levels. In [23] the parameter continuation for polynomial systems is used to remove junk points from $\widehat{W_i}$ to obtain the i-witness point set W_i. In the third section we give a modified algorithm using local dimension, Gröbner bases in the zero-dimensional case, and the homotopy function to compute the i-witness point set W_i. The breakup of the witness point set W_i into irreducible witness point sets is achieved using two algorithms (cf. [14],[22]). The first algorithm computes the points on the same irreducible component in the witness point set using path tracking techniques. The second algorithm computes a linear trace for each component which certifies the decomposition. In the fourth section we explain how to use the zero sum relation (cf. [7]) and the monodromy action on the algebraic variety to breakup W_i into irreducible witness point sets. In the last section we give examples and timings to compare the implementations of Bertini and Singular .

2 Witness Point Super Set

Definition 2.1. *Let $Z = V(f)$ be an affine algebraic variety in $\mathbb{C}^N$ of dimension d, and X be a pure i-dimensional component of Z. Let L_i be a generic linear space in $\mathbb{C}^N$ of dimension $N-i$. A finite set $\widehat{W_i} \subset \mathbb{C}^N$ is called i-witness point super set for X if*

$$X \cap L_i \subset \widehat{W_i} \subset Z \cap L_i \ .$$

The union $\widehat{W}$ of all i-witness point super sets is called a witness point super set for Z.

The following algorithm computes a witness point super set.

Remark 2.1. *With the notations of the algorithm the following facts prove its correctness and explain our modification:*

1. *The positive dimensional irreducible components of $V(F_1, ..., F_n)$ are the same as the positive dimensional irreducible components of $V(f_1, ..., f_N)$. Isolated points of $V(F_1, ..., F_n)$ are isolated points of $V(f_1, ..., f_N)$.*

2. *$V(f_1, ..., f_N)$ has no components of dimension smaller then $r := N - rank(f)$ (cf. [23]). Therefore the modified algorithm starts in dimension r.*

3. *Since $V(f_1, ..., f_N)$ is of dimension d, the witness point super sets in dimension greater than d are empty. Therefore the modified algorithm can stop at dimension d.*

4. *For $i = 0, 1, ..., d$, the sets $\widehat{W_i}$ are witness point super sets for the pure i-dimensional components of $V(F_1, ..., F_n)$ (cf. [15],[23],[24]).*

Algorithm 1 WITNESSPOINTSUPERSET

Input: $F = \{F_1, ..., F_n\} \subset \mathbb{C}[x_1, ..., x_N]$.

Output: $\{f_1, .., f_N\}, \{\widehat{W}_r, .., \widehat{W}_d\}, L$. $\{f_1, .., f_N\}$ a square system, $\widehat{W}_i$ a witness point super set corresponding to a pure i-dimensional component of $V(f_1, ..., f_N)$, L a set of linear polynomials defining a linear space of dimension $N - d$.

$f = \{f_1, ..., f_N\}$ reduction of $F = \{F_1, ..., F_n\}$ to a square system
(cf. [15],[18],[23]);
$d = dim(V(f_1, ..., f_N))$ (using Gröbner basis cf. [8],[11]);
$r = N - rank(f)$, rank(f) the rank of the Jacobian matrix of the system f at a
generic point;
$L = \{l_1, ..., l_d\}$ a set of d generic linear polynomials;
if $d = r$ **then**
compute $T_d = V(f_1, ..., f_N, l_1, ..., l_d)$ (using a solver based on triangular sets
cf. [8],[11]);
$\widehat{W}_d = \{(x_1, ..., x_N) \mid (x_1, ..., x_N) \in T_d,\ (x_1, ..., x_N) \in V(F)\}$;
return $\{f_1, ..., f_N\}$, $\{\widehat{W}_d\}$, L ;
else
for $i = r$ *to* d **do**
if $i = 0$ **then**
$\Omega_i(f)(x) = f$;
else

$$\Omega_i(f)(x, z_1, ..., z_i) =: \begin{pmatrix} f_1(x) + \sum_{j=1}^{i} \lambda_{1j} z_j \\ . \\ . \\ f_N(x) + \sum_{j=1}^{i} \lambda_{Nj} z_j \\ l_1 + z_1 \\ . \\ . \\ l_i + z_i \end{pmatrix}$$

$\lambda_{kj} \in \mathbb{C}$ generic, $k = 1, ..., N$, $j = 1, ..., i$;
for $i = d$ *to* r **do**
if $i = d$ **then**
compute $T_i = V(\Omega_i(f)(x, z_1, ..., z_i))$ (using a solver based on triangular
sets cf. [8],[11]);
else
compute $T_i = V(\Omega_i(f)(x, z_1, ..., z_i))$ (using a homotopy function with
$\Omega_{i+1}(f)(x, z_1, ..., z_i)$ as start system and S_{i+1} as start solution
set cf. [15],[18],[23]);
$\widehat{W}_i = \{(x_1, ..., x_N) \mid (x_1, ..., x_N, 0, ..., 0) \in T_i,\ (x_1, ..., x_N) \in V(F)\}$;
$S_i = T_i \setminus \{(x_1, ..., x_N, z_1, ..., z_i) \in T_i \mid z_1 = ... = z_i = 0\}$;
return $\{f_1, ..., f_N\}$, $\{\widehat{W}_r, ..., \widehat{W}_d\}$, L ;

5. *In [15],[18],[23] the cascade algorithm is used to compute* $\widehat{W_i}$. *It starts with* $i = N-1$ *to compute the witness point super sets* $\widehat{W_i}$. *It needs to define a start system* $G(x) = 0$ *for the homotopy continuation method (cf. [25],[26]), and to know its solutions. We use a Gröbner basis of the ideal defining* Z *to compute the dimension* d *of* Z, *then use the cascade algorithm which starts with* $i = d-1$. *We will show that we do not need to define a start system.*

We will illustrate our modifications of the algorithm by an example.

Example 2.1. *Let* X *be the algebraic variety defined by the polynomial system*

$$f(x,y,z) = \begin{pmatrix} (x^3+z)(x^2-y) \\ (x^3+y)(x^2-z) \\ (x^3+z)(x^3+y)(z^2-y) \end{pmatrix}.$$

The dimension of $X \subset \mathbb{C}^3$ *is 1.*
The algorithm in ([15],[23]) starts at level 2, *while the modified algorithm starts at level* 1 *to compute the witness point super set* $\widehat{W}$ *for the algebraic variety* X *as follows.*

- $L = \{l_1 = x+y+z-1\}$ *the set of* 1 *generic linear polynomials;*
- $$\Omega_0(f)(x,y,z) := f(x,y,z) = \begin{pmatrix} (x^3+z)(x^2-y) \\ (x^3+y)(x^2-z) \\ (x^3+z)(x^3+y)(z^2-y) \end{pmatrix};$$
- $\lambda_{11} := 1, \lambda_{12} := 5, \lambda_{13} := 18,$
  $$\Omega_1(f)(x,y,z,z_1) := \begin{pmatrix} (x^3+z)(x^2-y)+z_1 \\ (x^3+y)(x^2-z)+5z_1 \\ (x^3+z)(x^3+y)(z^2-y)+18z_1 \\ x+y+z-1+z_1 \end{pmatrix};$$
- *compute* $T_1 = \{t_1, ..., t_{29}\} \subset \mathbb{C}^4$ *the set of solutions of* $\Omega_1(f)(x,y,z,z_1)$ *using the library "solve.lib" in* Singular *;*
- $\widehat{W_1} = \{w_1, ..., w_7 \mid \exists t_i \in T_1 : t_i = (w_i, 0),\ i = 1, ..., 7\} \subset \mathbb{C}^3$ *the 1-witness point super set corresponding to the pure 1-dimensional component of* X*;*
- $S_1 = T_1 \setminus \{(x,y,z,z_1) \in T_1 \mid z_1 = 0\}$*;*

- *using the homotopy function technique, (implemented in* Bertini *),*

$$T_0 = V(t.\Omega_1(f)(x,y,z,z_1) + (1-t).\begin{pmatrix} \Omega_0(f)(x,y,z) \\ z_1 \end{pmatrix}),$$

  with the start system $\Omega_1(f)(x,y,z,z_1)$ and the start solution set S_1 as t goes from 1 to 0.

- $\widehat{W_0} = T_0 \subset \mathbb{C}^3$ *the 0-witness point super set corresponding to the pure 0-dimensional component of X;*

- $\widehat{W} = \{\widehat{W_0}, \widehat{W_1}\}$ *the witness point super set for X.*

We note that we did not need to define a start system (with given solutions) to compute the witness point super set.

3 Computation Witness Point Set

The witness point super set $\widehat{W_i}$ is a union of an i-witness point set W_i and a junk point set J_i (cf. [15],[18],[23]),

$$\widehat{W_i} = W_i \cup J_i, \;\; W_i \subset X_i \; and \; J_i \subset \cup_{j>i} X_j \; for \; i = 0,1,...,d.$$

We use Gröbner bases, triangular sets, local dimension and homotopy continuation method in the algorithm below to remove the points of J_i from $\widehat{W_i}$ as follows.

Proof (The correctness of the Algorithm 2).
The witness point super set $\widehat{W_i}$ is the union of points on the i-dimensional component and points on components of dimension greater then i. $\widehat{W_d}$ has no junk points, i.e. $W_d := \widehat{W_d}$. From the definition of the witness point set it follows that $s_d := \sharp W_d$ is the degree of the d-dimensional component of $V(f_1,...,f_n)$. The witness point super sets are computed numerically, that means $w \in \widehat{W_i}$ is an approximate value of a point v on X. Let $Z \subset \mathbb{C}^N \times \mathbb{C}^N$ be the algebraic variety defined by the polynomial system $\{f_1 - t_1, ..., f_N - t_N\}$, $y := (y_1,...,y_N) \in \mathbb{C}^N$ with $\|y\| \leq 10^{-16}$. Define the map $\varphi : Z \subset \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$ by $\varphi(x,y) = y$. Then we have

$$Z_{\varphi(v,0)} = V(f_1,...,f_N) \subset \mathbb{C}^N \; and$$

$^\dagger w$ is the numerical approximate solution of the system $f = \{f_1,...,f_n\}$, i.e. we consider $f(w) = 0$ numerically.

Algorithm 2 WITNESSPOINTSET

Input: $\{f_1,..,f_N\} \subset \mathbb{C}[x_1,..,x_N]$, $\{\widehat{W}_r,..,\widehat{W}_d\}$ a list of witness point super sets, $L=\{l_1,..,l_d\}$ a set of generic linear polynomials (Output of `Algorithm 1`).

Output: $\{f_1,...,f_N\}$, $\{W_r,..,W_d\}$, $L=\{l_1,...,l_d\}$. W_i a witness point set corresponding to a pure i-dimensional component of $V(f_1,...,f_N)$.

  $W_d=\widehat{W}_d$, $s_d=\sharp W_d$;
  for $i=d-1$ *to* r **do**
    $W_i=\widehat{W}_i$;
    for each point $w\in W_i$ **do**
      compute $t=dim_w Z$ for $Z=V(f_1-f_1(w),...,f_N-f_N(w))$ (using a Gröbner
        basis cf. [8],[11]);
      if $t>i$ **then**
        $W_i=W_i\setminus\{w\}$;
    for each point† $w\in W_i$ **do**
      if $i=0$ **then**
        choose $A\subset\mathbb{C}^{d\times N}$ a generic matrix and a generic $\epsilon\in\mathbb{C}^N$, $\|\epsilon\|<10^{-16}$;
        compute $S=V(\{f_1,...,f_N,A(x-w)\})$, $T=V(\{f_1,...,f_N,A(x-w-\epsilon)\})$
          (using a solver based on triangular sets cf. [8],[11]);
        if $\sharp S=\sharp T$ **then**
          $W_i=W_i\setminus\{w\}$;
      for $j=i+1$ *to* d **do**
        choose $A\subset\mathbb{C}^{j\times N}$ a generic matrix;
        if $j=d$ **then**
          compute $S=V(\{f_1,...,f_N,A(x-w)\})$ (using a solver based on
            triangular sets cf. [8],[11]);
          if $\sharp S=s_d$ **then**
            $W_i=W_i\setminus\{w\}$;
        else
          compute $S=V(\{f_1,...,f_N,A(x-w)\})$ (using a homotopy function with
            the start system $\{f_1,..,f_N,l_1,..,l_j\}$ and the start solution W_j cf.
            [23]);
          if $w\in S$ **then**
            $W_i=W_i\setminus\{w\}$;
  return $\{f_1,...,f_N\},\{W_r,...,W_d\},L$;

$$Z_{\varphi(x,f_1(w),...,f_N(w))} = V(f_1 - f_1(w), ..., f_N - f_N(w)) \subset \mathbb{C}^N.$$

It follows (cf. [9], proposition 3.4)

$$t := dim_w V(f_1 - f_1(w), ..., f_N - f_N(w)) \leq dim_v V(f_1, ..., f_N)).$$

If $t > i$, then w must be the approximate value of a point v on a component of dimension greater then i. That means that $w \in J_i$.
If $i = 0$, i.e. $w \in \widehat{W_0}$, then an $(N-d)$-dimensional generic linear space $V(A(x-w))$ meets the algebraic variety $V(f_1, ..., f_N)$ in a finite set S. If the $(N-d)$-dimensional generic linear space $V(A(x-w-\epsilon))$ passing through a neighborhood of w meets $V(f_1, ..., f_N)$ in a set T of the same cardinality, then there exists a neighborhood U of w such that $U \cap X \setminus \{w\} \neq \emptyset$. This implies that w is not an isolated point in $V(f_1, ..., f_N)$, i.e. w is on a component of positive dimension. This implies that $w \in J_i$. In case of $i > 0$ the test whether w is on a component of dimension $j \in \{d, d-1, ..., i+1\}$ is as follows. If $j = d$, the degree of the pure d-dimensional component is s_d. The d-dimensional generic linear space $V(A(x-w)^T)$ through w meets $V(f_1, ..., f_N)$ in a finite set S of cardinality greater or equal to s_d. If $\sharp S = s_d$, then w is on the pure d-dimensional component. It implies that $w \in J_i$. If $j < d$, we use the homotopy function to remove the junk points (cf. [23]). □

4 Partition Witness Point Sets

In this section we show that the monodromy action on an algebraic variety Z and the zero sum relation are sufficient to find the breakup of the k-witness point set W_k into irreducible k-witness point sets. We present here a modified version of the algorithms described in ([14],[22]).

Let Z be a pure k-dimensional algebraic variety in $\mathbb{C}^N$, and $Z = \cup_{i=1}^{r} Z_i$ be the irreducible decomposition of Z. Let $\pi : \mathbb{C}^N \longrightarrow \mathbb{C}^k$ be a generic projection and let $l \subset \mathbb{C}^k$ be a general line. Consider

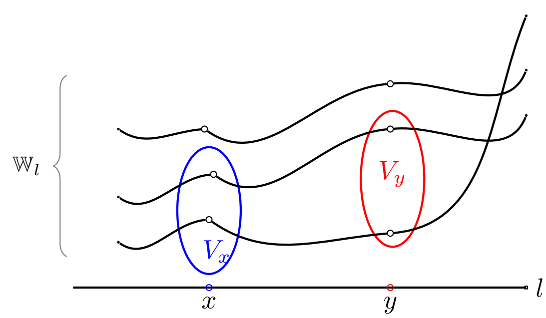
- $\mathbb{W}_l := \pi^{-1}(l) \cap Z$ a set of r different curves in $\mathbb{C}^N$.
- U the non-empty open subset of l consisting of all points $x \in l$ with $\pi^{-1}(x)$ transversal to Z.
- $W := \pi^{-1}(x) \cap Z$ for a generic element $x \in U$, and V a non-empty subset of W.
- $W_i := \pi^{-1}(x) \cap Z_i$ for an irreducible k-dimensional component Z_i of Z.
- $\lambda : \mathbb{C}^N \longrightarrow \mathbb{C}$ a linear function, one-to-one on W.

- For $y \in U$, let V_y be a subset of $\pi^{-1}(y) \cap Z$ defined by

$$V_y := \{z \mid z \textit{ on a curve in } \mathbb{W}_l \textit{ through a point of } V\}.$$

We define a function $s : U \longrightarrow \mathbb{C}$ by

$$s(y) = \sum_{z \in V_y} \lambda(z),$$



Theorem 4.1. *Let l, U, W, V, W_i for $i = 1, ..., r$, and the functions λ, s be as above.*
If the function s is continuous and $V \cap W_i \neq \emptyset$ for some $i \in \{1, ..., r\}$, then $W_i \subseteq V$.

Before proving the theorem we illustrate it by an example.

Example 4.1. *Let Z be the curve in $\mathbb{C}^2$ defined by the polynomial $f(x, y) = (x^2 + y^2 - 5)(x - 2y - 3)$. Let L_1 be the line in $\mathbb{C}^2$ defined by the polynomial $l_1 = x + y - 3$. We define a homotopy function :*

$$h(t, x(t), y(t)) := \begin{pmatrix} \alpha(t) \\ f(x(t), y(t)) \end{pmatrix}.$$

$$\alpha(t) = (1 - t)l_0 + tl_1 = x + y - 2t - 1, \textit{ where } l_0 = x + y - 1 \ .$$

Then with conditions above $\alpha(t)$ maps a point in $L_1 \cap Z$ to a point in $L_0 \cap Z$ as t goes from 1 to 0, L_0 the line defined by l_0.

Proof (of Theorem 4.1). Assume that $W_i \nsubseteq V$. Since $W_i \cap V \neq \emptyset$, then there are $a, b \in W_i$ such that a is not in V and b in V. Let $a_1, ..., a_r$ denote the points of the set $V \setminus \{b\}$. By Corollary 3.5 in [14] there is a loop α in the fundamental group $\pi_1(U, \pi^{-1}(x))$ with $\alpha(0) = \alpha(1)$ which takes a_j to a_j for all j=1,...,r, and interchanges a and b.

Since α is a continuous loop and $s : U \longrightarrow \mathbb{C}$ is continuous, the composition $s \circ \alpha : [0,1] \longrightarrow \mathbb{C}$ is continuous and

$$s(\alpha(1)) = s(\alpha(0))$$

$$\lambda(a) + \sum_{j=1}^{r} \lambda(a_j) = \lambda(b) + \sum_{j=1}^{r} \lambda(a_j),$$

as t goes from 1 to 0. This implies that $\lambda(a) = \lambda(b)$. But this contradicts the fact that λ is one-to-one on W. Thus W_i is a subset of V. □

Example 4.2. *Let Z be the curve in $\mathbb{C}^2$ defined be the polynomial $f(x,y) = (y-x)(y-2x)(y-3x)$, and $Z = Z_1 \cup Z_2 \cup Z_3$ be the irreducible decomposition. Let $\pi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be the projection given by $\pi(x,y) = x$, and $\lambda : \mathbb{C}^2 \longrightarrow \mathbb{C}$, $\lambda(x,y) = y$.*
Note that the restriction of π to Z, π_Z is proper and generically three-to-one with degree 3 equal to the degree of Z. λ is one-to-one on the fiber $\pi^{-1}(y) = \{(x,x),(x,2x),(x,3x)\}$. Let L be the line defined by the linear polynomial $l(x,y) = x+y-2$. L intersects Z in the finite set $W := \{(1,1),(\frac{2}{3},\frac{4}{3}),(\frac{1}{2},\frac{3}{2})\}$. Let $V := \{(1,1),(\frac{2}{3},\frac{4}{3})\} \subset W$. The function $\sum_{v \in V} \lambda(v)$ given by $\lambda(x,x) + \lambda(x,2x) = x + 2x = 3x$ is continuous. By the theorem above if an irreducible 1-witness point set W_1 contains $\{(1,1)\}$ or $\{(\frac{2}{3},\frac{4}{3})\}$, then W_1 is a subset of V.

Now we will explain our modification of the algorithm to compute the irreducible witness point sets.
Let $Z_k = \cup_{i=1}^{r} Z_{ki}$ be the union of the irreducible k-dimensional components of the algebraic variety $Z = V(f_1, ..., f_n)$ and L_k be the linear space in $\mathbb{C}^N$ defined by k generic linear polynomials

$$l_j = c_{j0} + c_{j1}x_1 + ... + c_{jN}x_N.$$

for $j = 1, .., k$ and $i = 0, 1, ..., N$, $c_{ij} \in \mathbb{C}$.
We use the generic linear space L_k to define a projection
$\pi : \mathbb{C}^N \longrightarrow \mathbb{C}^{k+1}$, $\pi(x_1, ..., x_N) := (z_1, ..., z_k, z_{k+1})$ as follows:

$$\begin{pmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_N \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ . \\ . \\ . \\ z_k \\ z_{k+1} \end{pmatrix} := \begin{pmatrix} c_{11} & c_{12} & . & . & c_{1N} \\ c_{21} & c_{22} & . & . & c_{1N} \\ . & . & . & . & . \\ . & . & . & . & . \\ c_{k1} & c_{k2} & . & . & c_{kN} \\ p_1 & p_2 & . & . & p_N \end{pmatrix} . \begin{pmatrix} x_1 \\ . \\ . \\ . \\ . \\ x_N \end{pmatrix},$$

$p_1, ..., p_N \in \mathbb{C}$ randomly chosen.
Set $\lambda(x_1, ..., x_N) := z_{k+1}$ and $l := V(z_1, ..., z_{k-1}) \subset \mathbb{C}^k$ the coordinate axis z_k

as in the theorem above. Let $L_{k,y}$ be the linear space defined by the linear polynomials $l_1, ..., l_{k-1}$ and $l_{k,y} := y + c_{k1}x_1 + ... + c_{kN}x_N$. Let $W_y := L_{k,y} \cap Z_k$ be the k-witness point set. For $y = c_{k0}$, we fix a non-empty subset $V = V_y \subset W_y$. In general let V_y be the subset of W_y consisting of all points which are on a curve in $\pi^{-1}(l) \cap Z_k$ through a point of V. To compute V_y we use the homotopy function

$$H(x(t), t) = (1-t).\begin{pmatrix} l_1 \\ . \\ . \\ l_k \\ f_1 \\ . \\ . \\ f_n \end{pmatrix} + t.\begin{pmatrix} l_1 \\ . \\ . \\ l_k - c_{k0} + y \\ f_1 \\ . \\ . \\ f_n \end{pmatrix}$$

as t goes from 1 to 0 using V as start system. Note that $\sharp V_y = \sharp V$.

Define the function $s : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$s(y) := \sum_{(x_1,...,x_N) \in V_y} \lambda(x_1, ..., x_N).$$

To test the linearity of s, we take three values of y in $\mathbb{C}$, say a, b, c.
If there exist $A, B \in \mathbb{C}$ such that

$$(s(a) = Aa + B, s(b) = Ab + B) \Longrightarrow s(c) = Ac + B, \tag{4.1}$$

then s is linear.

So far this is the approach which can be found in [14]. We now explain a modification.
The condition (4.1) of the linearity above is equivalent to the following equation

$$s(a)(b-c) + s(b)(c-a) + s(c)(a-b) = 0. \tag{4.2}$$

If $W_{kj} \cap V_a \neq \emptyset$ for some $j \in \{1, ..., r\}$ and the condition (4.1) is true, then $W_{kj} \subseteq V_a$ (Theorem 4.1). Let

$$Z(y) := \{z = \sum_{t=1}^{N} p_t v_t \mid v = (v_1, ..., v_N) \in V_y, p = (p_1, ..., p_N) \in \mathbb{C}^N\}.$$

Then

$$s(y) = \sum_{v \in V_y} \lambda(v) = \sum_{v \in V_y} (\sum_{t=1}^{N} p_t v_t) = \sum_{z \in Z(y)} z.$$

The continuation of the homotopy function implies that the i-th points in the sets V_a, V_b and V_c are on the same irreducible component. Let $V_a := \{v_1, ..., v_m\}$, $V_b := \{\overline{v}_1, ..., \overline{v}_m\}$ and $V_c := \{\hat{v}_1, ..., \hat{v}_m\}$ be the sets computed by using the homotopy function above . Let $Z(a) := \{a_1, ..., a_m\}$, $Z(b) := \{b_1, ..., b_m\}$ and $Z(c) := \{c_1, ..., c_m\}$ be the sets corresponding to the set V_a, V_b and V_c respectively.

From (4.2) we obtain an equivalent condition to (4.1)

$$(b-c)\sum_{i=1}^{m} a_i + (c-a)\sum_{i=1}^{m} b_i + (a-b)\sum_{i=1}^{m} c_i = 0. \tag{4.3}$$

The condition (4.3) is called **zero sum relation** (cf. [7]) of a given subset $V_a \subseteq W$ denoted by $ZSR(V_a)$. The sets V_a, V_b and V_c have distinct points and the same cardinality m, then obviously

$$ZSR(V_a) = \sum_{a_i \in V_a} ZSR(\{a_i\}). \tag{4.4}$$

where $ZSR(\{a_i\}) = (b-c)a_i + (c-a)b_i + (a-b)c_i$ is defined as the zero sum relation of a given point in V_a.

The following algorithm computes irreducible witness point sets. The correctness of the Algorithm 3 follows from the Theorem (4.1).
We give an example of a pure 2-dimensional variety Z which is a union of two 2-dimensional irreducible components Z_1 *and* Z_2. Z_1 is of degree three and Z_2 is of degree two. The 2-witness point set W for Z is given as a finite subset of Z consisting of five points $\{w_1, w_2, w_3, w_4, w_5\}$. Z_1 should contain three points $W_1 := \{w_1, w_2, w_3\}$ and the remaining points $W_2 := \{w_4, w_5\}$ are on Z_2. The algorithms (cf. [14],[22]) use the homotopy function at least nine times to breakup W into W_1 *and* W_2. We will show below that we do not need more than five times to use the homotopy function to breakup W into W_1 *and* W_2.

Example 4.3. .
Let Z be the algebraic variety of dimension two in $\mathbb{C}^3$ defined by the polynomial $f(x, y, z) = (x^3 + z)(x^2 - y)$. Let L be the linear space of dimension one in $\mathbb{C}^3$ defined by the linear equations $l_1 = 4x+7y+2z+6$, $l_2 = 5x+7y+3z+6$. Then $W := L \cap Z = \{w_1, w_2, w_3, w_4, w_5\}$, where $w_1 = (1, -1.1428571429, -1), w_2 = (0, -0.8571428571, 0)$,

‡the $i-th$ point in R corresponds to the $i-th$ point in W_a;
§smallest subset with respect to the cardinality.

Algorithm 3 IrrWitnessPointSet

Input: $\{f_1, ..., f_N\} \subset \mathbb{C}[x_1, ..., x_N]$, $\{W_r, ..., W_d\}$, a list of witness point sets, $L = \{l_1, ..., l_d\}$ a set of generic linear polynomials (Output of `Algorithm 2`). Where $W_k = \{w_1, ..., w_{m_k}\}$ are witness point sets for a pure k-dimensional component Z_k of $Z = V(f_1, ..., f_N)$, $k = r, ..., d$.

Output: $\{\{W_{r1}, ..., W_{rt_r}\}, ..., \{W_{d1}, ..., W_{dt_d}\}\}$, W_{kr_k} irreducible witness point sets corresponding to a k-dimensional irreducible component Z_{kr_k} of Z_k.

for $k = r$ *to* d **do**
  a:=c_{k0};
  define L_{ka} to be the linear space defined by the subset $\{l_1, ..., l_k\} \subset L$;
  choose $b, c \in \mathbb{C}$ generic, define L_{kb}, L_{kc} as above;
  $W_a = W_k$, $W_b = \emptyset$, $W_c = \emptyset$, $R = \emptyset$;
  choose $p_1, ..., p_N \in \mathbb{C}$;
  for $i = 1$ *to* m_k **do**
    compute $\{v_i\} \subset Z \cap L_{k,b}$ and $\{\widehat{v}_i\} \subset Z \cap L_{k,c}$ (using the homotopy function with $\{f_1, ..., f_N, l_1, ..., l_{k-1}, l_{k,a}\}$ as start system and $\{w_i\}$ as start solution);
    compute the zero sum relation of $\{w_i\}$;

$$r_i = (a-b)(\sum_{j=1}^{N} p_j \widehat{v}_{ij}) + (b-c)(\sum_{j=1}^{N} p_t w_{ij}) + (c-a)(\sum_{j=1}^{N} p_t v_{ij});$$

    $R = R \cup \{r_i\}^{\ddagger}$;
  $t_k = 0$;
  while $R \neq \emptyset$ **do**
    if $\sum_{t \in T} t = 0$ and T is a smallest subset§ of R **then**
      $t_k = t_k + 1$;
      $W_{kt_k} \subset W_a$ the points corresponding of the points of T;
      $R = R \setminus T$;
return $\{\{W_{r1}, ..., W_{rt_r}\}, ..., \{W_{d1}, ..., W_{dt_d}\}\}$;

$w_3 = (-0.1428571429 + i * 0.9147320339, -0.8163265306 - i * 0.2613520097, 0.1428571429 - i * 0.9147320339)$,
$w_4 = (-1, -0.5714285714, 1)$,
$w_5 = (-0.1428571429 - i * 0.9147320339, -0.8163265306 + i * 0.2613520097, 0.1428571429 + i * 0.9147320339)$.

We now illustrate the Algorithm 3:

- *Use the linear space L_1 to define the linear projection $\pi : \mathbb{C}^3 \longrightarrow \mathbb{C}^3$ as follows*

$$\pi(x, y, z) := \begin{pmatrix} 4 & 7 & 2 \\ 5 & 7 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (4x+7y+2z, 5x+7y+3z, x+2y+3z).$$

- *Define the linear space $L_{1,c}$ of dimension one in $\mathbb{C}^3$ by the linear equations $l_1 = 4x + 7y + 2z + 6$, $l_c = 5x + 7y + 3z + c$, where c is generically chosen in $\mathbb{C}$. Then*

$$\pi_{Z \cap L_{1,c}}(x, y) = (-6, -c, x + 2y + 3z).$$

- *Define the linear function $\lambda : \mathbb{C}^2 \longrightarrow \mathbb{C}$ by $\lambda(x, y, z) := x + 2y + 3z$.*

- *For $a = 6$, let $V_1 = V_a := \{w_{11} = (1, -1.1428571429, -1)\} \subset W$, $L_{1,a} :=$ L the linear space defined by $l_1 = 4x + 7y + 2z + 6$, $l_a = 5x + 7y + 3z + 6$. Then*[¶] $Z(a) = \{\sum_{v \in V_a} \lambda(v) = w_{11}[1] + 2(w_{11}[2]) + 3(w_{11}[3])\} = \{-4.2857142858\}$.

- *Let $b = 9$, $L_{1,b}$ the linear space defined by $l_1 = 4x + 7y + 2z + 6$, $l_b = 5x + 7y + 3z + 9$. Compute $V_b := (tL_{1,a} + (1-t)L_{1,b}) \cap Z = \{w_{12} = (1.671699881657157, -0.4776285376163331, -4.671699881657164)\}$ as t goes from 1 to 0, using V_a as a start solution. $Z(b) = \{w_{12}[1] + 2(w_{12}[2]) + 3(w_{12}[3])\} = \{-13.2986568385470012\}$.*

- *Let $c = 63$, $L_{1,c}$ the linear space defined by $l_1 = 4x + 7y + 2z + 6$, $l_c = 5x + 7y + 3z + 63$. Compute $V_c := (tL_{1,a} + (1-t)L_{1,c}) \cap Z = \{w_{13} = (3.935100643260828, 14.30425695906836, -60.93510064326094)\}$ as t goes*
  from 1 to 0, using V_a as a start solution. $Z(c) = \{w_{13}[1] + 2(w_{13}[2]) + 3(w_{13}[3])\} = \{-150.261687368385272\}$.

[¶] we use the notation $w_{ij} = (w_{ij}[1], w_{ij}[2], w_{ij}[3])$ for i=1,..,5, j=1,2,3.

zero sum relation of $V_1 = \{(1, -1.1428571429, -1)\}$*:*

$$r_1 := \sum_{a\in Z(a)} (b-c) + \sum_{b\in Z(b)} (c-a) + \sum_{c\in Z(c)} (a-b) =$$

$$= -75.8098062588232524.$$

The zero sum relation set of $V_1 = \{(1, -1.1428571429, -1)\}$ *is* $R_1 := \{r_1 = -75.8098062588232524\}$.

- *Let* $a = 6$, $V_a := \{w_{11} = (0, -0.8571428571, 0)\} \subset W$, $L_{1,a} := L$ *the linear space defined by* $l_1 = 4x + 7y + 2z + 6$, $l_a = 5x + 7y + 3z + 6$. *Then* $Z(a) = \{\sum_{v\in V_a} \lambda(v) = w_{11}[1] + 2(w_{11}[2]) + 3(w_{11}[3])\} = \{-1.7142857142\}$.

- *Let* $b = 9$, $L_{1,b}$ *the linear space defined by* $l_1 = 4x + 7y + 2z + 6$, $l_b = 5x + 7y + 3z + 9$. *Compute* $V_b := (tL_{1,a} + (1-t)L_{1,b}) \cap Z = \{w_{12} = (-0.8358499408285809 + i * 1.046869318849985, 0.2388142688081706 - i * 0.2991055196714253, -2.164150059171436 - i * 1.046869318849981)\}$ *as* t *goes from 1 to 0, using* V_a *as a start solution.* $Z(b) = \{w_{12}[1] + 2(w_{12}[2]) + 3(w_{12}[3])\} = \{-6.8506715807265477 - i*2.6919496770428086\}$.

- *Let* $c = 63$, $L_{1,c}$ *the linear space defined by* $l_1 = 4x + 7y + 2z + 6$, $l_c = 5x + 7y + 3z + 63$. *Compute* $V_c := (tL_{1,a} + (1-t)L_{1,c}) \cap Z =$
  $\{w_{13} = (-1.967550321630417 + i * 3.257877039491183,$
  $15.99072866332302 - i * 0.9308220112831772,$
  $-55.03244967836969 - i * 3.257877039491242);\}$ *as* t *goes from 1 to 0, using* V_a *as a start solution.* $Z(c) = \{w_{13}[1] + 2(w_{13}[2]) + 3(w_{13}[3])\} = \{-135.083442030093447 - i * 8.3773981015488974\}$.

zero sum relation of $V_2 = \{(0, -0.8571428571, 0)\}$*:*

$$r_2 := \sum_{a\in Z(a)} (b-c) + \sum_{b\in Z(b)} (c-a) + \sum_{c\in Z(c)} (a-b) =$$

$$= 107.3334745556671221 - i * 128.308937286793398.$$

The zero sum relation set of $V_2 = \{(0, -0.8571428571, 0)\}$ *is*
$R_2 := \{r_2 = 107.3334745556671221 - i * 128.308937286793398\}$.

- *For the other points* $V_3 = \{w_3\}, V_4 = \{w_4\}$ *and* $V_5 = \{w_5\}$, *we found the zero sum relations:*
  $R_3 := \{r_3 = -9.38237104997583366 + i * 127.0170767088\}$,
  $R_4 := \{r_4 = -31.5236682999307779 + i * 128.3089372867945956\}$ *and*
  $R_5 := \{r_5 = 9.382371038077068 - i * 127.0170767088\}$.

- *The set of zero sum relation for all points of W is $R = \cup_{j=1}^{5} R_j = \{r_1, r_2, r_3,$
  $r_4, r_5\}$, where i-th point in W corresponds i-th point in R.*
- *Find the smallest subset T of R with $\sum_{t \in T} t = 0$, which corresponds an irreducible witness point set of W. Then we get $T_1 = \{r_3, r_5\}$, $T_2 = \{r_1, r_2, r_4\}$ corresponding to the irreducible witness point sets $W_1 = \{w_3, w_5\}$, $W_2 = \{w_1, w_2, w_4\}$ respectively.*

Remark 4.1. *The points of a witness point set are computed approximately by using the homotopy continuation method. Therefore the result of the zero sum relation is only almost zero.*

5 Examples and timings with Singular and Bertini

In this section we provide examples with timings of the algorithms `WitnessPointSuperSet`, `WitnessPointSet`, and `IrrWitnessPointSet` implemented in Singular to compute‖ the numerical decomposition of a given algebraic variety defined by a polynomial system and compare them with the results of Bertini . Each step of the numerical decomposition is parallelizable. For our comparisons we did not use the parallel version of Bertini .

We tested two versions of the implementation in Bertini using the cascade algorithm (cf. [15]) and using the regenerative cascade (cf. [16]) algorithm. For the timings we used the 32-bit version of Singular 3-1-1 (cf. [8]) and Bertini 1.2 (cf. [3]) on an Intel® Core(TM)2 Duo CPU P8400 @ 2.26 GHz 2.27 GHz, 4 GB RAM under the Kubuntu Linux operating system.

Let Z be the algebraic variety defined by the following polynomial system:

Example 5.1. *(cf. [18]).*

$$f(x,y,z) = \begin{pmatrix} (y-x^2)(x^2+y^2+z^2-1)(x-\frac{1}{2}) \\ (z-x^3)(x^2+y^2+z^2-1)(y-\frac{1}{2}) \\ (y-x^2)(z-x^3)(x^2+y^2+z^2-1)(z-\frac{1}{2}) \end{pmatrix}$$

Example 5.2. *(cf. [23],Example 13.6.4).*

$$f(x,y,z) = \begin{pmatrix} x(y^2-x^3)(x-1) \\ x(y^2-x^3)(y-2)(3x+y) \end{pmatrix}$$

‖The Singular implementation uses Bertini to compute the solutions of the homotopy function.

Example 5.3.

$$f(x,y,z) = \begin{pmatrix} (x^3+z)(x^2-y) \\ (x^3+y)(x^2-z) \\ (x^3+z)(x^3+y)(z^2-y) \end{pmatrix}$$

Example 5.4.

$$f(x,y,z) = \begin{pmatrix} x(y^2-x^3)(x-1) \\ x(3x+y)(y^2-x^3)(y-2) \\ x(y^2-x^3)(x^2-y) \end{pmatrix}$$

Example 5.5.

$$f(x,y,z) = \begin{pmatrix} (x-1)((x^3+z)+(x^2-y)) \\ (x^3+z)(x^2-y) \\ (x^3+z)(x^2-1) \end{pmatrix}$$

Example 5.6.

$$f(x,y,z) = \begin{pmatrix} (y-x^2)(x^2+y^2+z^2-1)(x-\frac{1}{2})+x^5 \\ (z-x^3)(x^2+y^2+z^2-1)(y-\frac{1}{2})+y^4 \\ (y-x^2)(z-x^3)(x^2+y^2+z^2-1)(z-\frac{1}{2})+z^6 \end{pmatrix}$$

Example 5.7.

$$f(x,y,z) = \begin{pmatrix} x(y^2-x^3)(x-1)+y^2 \\ x(y^2-x^3)(y-2)(3x+y)+x^3 \end{pmatrix}$$

Example 5.8.

$$f(x,y,z) = \begin{pmatrix} (x^3+z)(x^2-y)+x^4 \\ (x^3+y)(x^2-z)+y^3 \\ (x^3+z)(x^3+y)(z^2-y)+z^5 \end{pmatrix}$$

Example 5.9.

$$f(x,y) = \begin{pmatrix} -3568891411860300072x^5 + 1948764938x^4 + \\ 3568891411860300072x^2y^2 - 1948764938xy^2 \\ \\ -5105200242937540320x^5y - 1701733414312513440x^4y^2 + \\ 11692589628x^5 + 3897529876x^4y + 5105200242937540320x^2y^3 + \\ 1701733414312513440xy^4 - 11692589628x^2y^2 - 3897529876xy^3 \end{pmatrix}$$

Example 5.10.

$$f(x,y,z)=\begin{pmatrix} -356737285367005125x^5-92300457164036000x^3y+ \\ 1121648050080163317x^2z+290209720279281056yz \\ \\ -356737285367005125x^5+887060318883271500x^3z+ \\ 1121648050080163317x^2y-2789081819567309964yz \\ \\ -356737285367005125x^5z^2+356737285367005125x^5y+ \\ 887060318883271500x^3z^3-887060318883271500x^3yz+ \\ 1121648050080163317x^2z^3-1121648050080163317x^2yz- \\ 2789081819567309964z^4+2789081819567309964yz^2 \end{pmatrix}$$

Example 5.11.

$$f(x,y,z)=\begin{pmatrix} x^5y^2+2x^3y^4+xy^6+2x^3y^2z^2+2xy^4z^2+xy^2z^4-x^4y^2 \\ -2x^2y^4-y^6-x^5z-2x^3y^2z-xy^4z-2x^2y^2z^2-2y^4z^2- \\ 2x^3z^3-2xy^2z^3-y^2z^4-xz^5-3x^3y^2-3xy^4+x^4z+ \\ 2x^2y^2z+y^4z-3xy^2z^2+2x^2z^3+2y^2z^3+z^5+3x^2y^2+ \\ 3y^4+3x^3z+3xy^2z+3y^2z^2+3xz^3+2xy^2-3x^2z-3y^2z- \\ 3z^3-2y^2-2xz+2z \\ \\ x^6y+2x^4y^3+x^2y^5+2x^4yz^2+2x^2y^3z^2+x^2yz^4- \\ 5x^6-10x^4y^2-5x^2y^4-x^4yz-2x^2y^3z-y^5z-10x^4z^2- \\ 10x^2y^2z^2-2x^2yz^3-2y^3z^3-5x^2z^4-yz^5-3x^4y- \\ 3x^2y^3+5x^4z+10x^2y^2z+5y^4z-3x^2yz^2+10x^2z^3+ \\ 10y^2z^3+5z^5+15x^4+15x^2y^2+3x^2yz+3y^3z+15x^2z^2 \\ +3yz^3+2x^2y-15x^2z-15y^2z-15z^3-10x^2-2yz+10z \\ \\ x^6y^2z+2x^4y^4z+x^2y^6z+2x^4y^2z^3+2x^2y^4z^3+ \\ x^2y^2z^5-7x^6y^2-14x^4y^4-7x^2y^6-x^6z^2-17x^4y^2z^2- \\ 17x^2y^4z^2-y^6z^2-2x^4z^4-11x^2y^2z^4-2y^4z^4-x^2z^6- \\ y^2z^6+7x^6z+18x^4y^2z+18x^2y^4z+7y^6z+15x^4z^3+ \\ 27x^2y^2z^3+15y^4z^3+9x^2z^5+9y^2z^5+z^7+21x^4y^2+ \\ 21x^2y^4-4x^4z^2+13x^2y^2z^2-4y^4z^2-11x^2z^4-11y^2z^4- \\ 7z^6-21x^4z-40x^2y^2z-21y^4z-24x^2z^3-24y^2z^3-3z^5- \\ 14x^2y^2+19x^2z^2+19y^2z^2+21z^4+14x^2z+14y^2z+ \\ 2z^3-14z^2 \end{pmatrix}$$

Example 5.12.

$$f(x_1,x_2,x_3,x_4,x_5) = \begin{pmatrix} x_5^2 + x_1 + x_2 + x_3 + x_4 - x_5 - 4 \\ x_4^2 + x_1 + x_2 + x_3 - x_4 + x_5 - 4 \\ x_3^2 + x_1 + x_2 - x_3 + x_4 + x_5 - 4 \\ x_2^2 + x_1 - x_2 + x_3 + x_4 + x_5 - 4 \\ x_1^2 - x_1 + x_2 + x_3 + x_4 + x_5 - 4 \end{pmatrix}$$

Example 5.13.

$$f(a,b,c,d,e,f,g) = \begin{pmatrix} a^2 + 2de + 2cf + 2bg + a \\ 2ab + e^2 + 2df + 2cg + b \\ b^2 + 2ac + 2ef + 2dg + c \\ 2bc + 2ad + f^2 + 2eg + d \\ c^2 + 2bd + 2ae + 2fg + e \\ 2cd + 2be + 2af + g^2 + f \\ d^2 + 2ce + 2bf + 2ag + g \end{pmatrix}$$

Example 5.14. *cyclic 4-roots problem.(cf.[5],[6]).*

Example 5.15. *cyclic 5-roots problem.(cf.[5],[6]).*

Example 5.16. *cyclic 6-roots problem.(cf.[5],[6]).*

Example 5.17. *cyclic 7-roots problem.(cf.[5],[6]).*

Example 5.18. *cyclic 8-roots problem.(cf.[5],[6]).*

Example 5.19.

$$f(x_{11},x_{12},x_{13},x_{14},x_{15},x_{21},x_{22},x_{23},x_{24},x_{25},x_{31},x_{32},x_{33},x_{34},x_{35}) =$$

$$= \begin{pmatrix} -x_{12}x_{21} + x_{11}x_{22} \\ -x_{13}x_{22} + x_{12}x_{23} \\ -x_{14}x_{23} + x_{13}x_{24} \\ -x_{15}x_{24} + x_{14}x_{25} \\ -x_{22}x_{31} + x_{21}x_{32} \\ -x_{23}x_{32} + x_{22}x_{33} \\ -x_{24}x_{33} + x_{23}x_{34} \\ -x_{25}x_{34} + x_{24}x_{35} \end{pmatrix}$$

Table 1 summarizes the results of the timings to compute the numerical decomposition**.

Remark 5.1. *The timings in the following table show that for an increasing number of variables the original method of (cf.[14],[15],[18],[22],[23]) becomes more efficient. One reason is that the computation of triangular sets which is used in* SINGULAR *for solving polynomial systems is expensive in this case. Therefore the Algorithm 1, Algorithm 2 become slow in this situation. This is not true for the Algorithm 3.*
Replacing the solving of polynomial systems using triangular sets by homotopy function methods but keeping the computation of the dimension and starting at this dimension is more efficient also in case of a large number of variables.

Example	BERTINI	BERTINI (re)	SINGULAR
5.1	134.45s	39s	36.07
5.2	3.08s	2.5s	1.49s
5.3	1min 21.28s	27.4s	4.02s
5.4	18.56s	2.7s	1.77s
5.5	15.36s	8.6s	1.29s
5.6	4min 13s	15min 2s	2min 27s
5.7	1.83s	1.6s	0.39s
5.8	3min 29s	10min 43s	1.69s
5.9	16s	7s	2s
5.10	2min 57s	28s	2min 35s
5.11	44min 56s	2min 37s	4min 3s
5.12	4.73s	6s	0.37s
5.13	5.84s	8s	1s
5.14	1.43s	4.3s	0.79s
5.15	3.54s	10s	0.57s
5.16	3min 23.26s	2min 29s	1.43s
5.17	2h 11min 57s	32min 17s	stopped after 5h
5.18	19h48min 17s	6h45min2s	stopped after 50h
5.19	1min 57s	51s	stopped after 3h

Table 1: Total running times for computing a numerical decomposition of the examples above

** (re) means using the regenerative cascade algorithm instead of the cascade algorithm

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