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Spectral Properties of Nonhomogenous Differential Equations with Spectral Parameter in the Boundary Condition

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Abstract

In this paper, using the boundary properties of the analytic functions we investigate the structure of the discrete spectrum of the boundary value problem

$$\begin{aligned} iy_1' + q_1(x)y_2 - \lambda y_1 &= \varphi_1(x) \\ -iy_2' + q_2(x)y_1 - \lambda y_2 &= \varphi_2(x), \qquad x \in R_+ \end{aligned} \tag{0.1}$$

and the condition

$$(a_1\lambda + b_1)y_2(0,\lambda) - (a_2\lambda + b_2)y_1(0,\lambda) = 0 \tag{0.2}$$

where $q_1, q_2, \varphi_1, \varphi_2$ are complex valued functions, $a_k \neq 0, b_k \neq 0$, $k = 1,2$ are complex constants and λ is a spectral parameter. In this article, we investigate the spectral singularities and eigenvalues of (0.1), (0.2) using the boundary uniqueness theorems of analytic functions. In particular, we prove that the boundary value problem (0.1), (0.2) has a finite number of spectral singularities and eigenvalues with finite multiplicities under the conditions,

$$\sup_{x\in R_+} [\quad | \quad \varphi_k(x) \mid \exp(\varepsilon x^{\delta})] < \infty, \;\; k = 1.2$$

$$\sup_{x\in R_+} [\quad | \quad q_k(x) \mid \exp(\varepsilon x^{\delta})] < \infty, \;\; k = 1.2$$

for some $\varepsilon > 0, \quad \frac{1}{2} < \delta < 1$.

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1 Introduction

Let us consider the boundary value problem the $L[\lambda]$ generated in Hilbert space of vector valued functions the $L^2(R_+, C_2)$ by the system

$$\begin{aligned} iy_1' + q_1(x)y_2 &= \lambda y_1, \\ -iy_2' + q_2(x)y_1 &= \lambda y_2, \qquad x \in R_+ \end{aligned}$$

and the spectral parameter dependent boundary condition

$$(a_1\lambda + b_1)y_2(0,\lambda) - (a_2\lambda + b_2)y_1(0,\lambda) = 0.$$

Under the condition

$$\sup_{x \in R_+} [\mid q_k(x) \mid \exp(\varepsilon x^{1+\delta})] < \infty, \, \delta > 0, \quad k = 1, 2$$

it is proved that the $L[\lambda]$ has finite number of eigenvalues and spectral singularities with finite multiplicities. Furthermore, the principal functions corresponding to the eigenvalues of the $L[\lambda]$ belong to the $L^2(R_+, C_2)$ and the principal functions corresponding to spectral singularities belong to a Hilbert space containing the $L^2(R_+, C_2)$ [9]. We now consider the operator the $L_1(\lambda)$ generated in

$$L^2(R_+, C_2) := \left\{ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \int_0^\infty \left\{ |f_1(x)|^2 + |f_2(x)|^2 \right\} dx < \infty \right\}$$

by the system

$$\begin{aligned} iy_1' + q_1(x)y_2 - \lambda y_1 = 0, \\ -iy_2' + q_2(x)y_1 - \lambda y_2 = 0, \qquad x \in R_+ \end{aligned} \tag{1.1}$$

and the spectral parameter-dependent boundary condition

$$(a_1\lambda + b_1)y_2(0,\lambda) - (a_2\lambda + b_2)y_1(0,\lambda) = 0 \tag{1.2}$$

where q_k, $k = 1, 2$, are complex valued functions λ is the spectral parameter, a_k, b_k are complex constants, $b_k \neq 0, k = 1, 2$. Moreover $|a_1|^2 + |a_2|^2 \neq 0$. Note that the spectral analysis of homogeneous Schrödinger, Strum-Liouville and Dirac equations with spectral singularities were studied in details in $[3-9]$.

2 The Solutions of Equation (1.1)

Let us suppose that

$$|q_k(x)| \leq c(1+x)^{-(1+\varepsilon)}, \quad k=1,2, \quad x \in R_+, \ \varepsilon > 0 \tag{2.1}$$

holds, where $c>0$ is a constant. The following results were given in [1]. Under the condition (2.1), the equation (1.1) has the following vector solutions

$$e^+(x,\lambda) = \begin{pmatrix} e_1^+(x,\lambda) \\ e_2^+(x,\lambda) \end{pmatrix} = \begin{pmatrix} \int\limits_0^\infty H_{12}(x,t)e^{i\lambda t}dt \\ e^{i\lambda x} + \int\limits_0^\infty H_{22}(x,t)e^{i\lambda t}dt \end{pmatrix} \tag{2.2}$$

for $\lambda \in \bar{\mathbf{C}}_+$ and

$$e^-(x,\lambda) = \begin{pmatrix} e_1^-(x,\lambda) \\ e_2^-(x,\lambda) \end{pmatrix} = \begin{pmatrix} e^{-i\lambda x} + \int\limits_0^\infty H_{11}(x,t)e^{-i\lambda t}dt \\ \int\limits_0^\infty H_{21}(x,t)e^{-i\lambda t}dt \end{pmatrix} \tag{2.3}$$

for $\lambda \in \bar{\mathbf{C}}_-$ where

$$\begin{aligned} C_+(x,\lambda) &= \{\lambda : \lambda \in C, \, Im\lambda \geq 0\} \\ C_-(x,\lambda) &= \{\lambda : \lambda \in C, \, Im\lambda \leq 0\} \end{aligned}$$

moreover the kernels $H_{kj}(x,t)$, $k,j=1,2$, satisfy the inequalities

$$|H_{kj}(x,t)| \leq c \sum_{n=1}^{2} \left| q_n\left(\frac{x+t}{2}\right)\right|, \tag{2.4}$$

where $c>0$ is a constant. Therefore the functions $e_k^+(x,\lambda)$ and $e_k^-(x,\lambda)$ $k=1,2$ are analytic with respect to λ in $\mathbf{C}_+$, $\mathbf{C}_-$ and continuous on $\bar{\mathbf{C}}_+$ and $\bar{\mathbf{C}}_-$, respectively. Moreover $e^+(x,\lambda)$ and $e^-(x,\lambda)$ satisfy the following asymptotic equalities([1])

$$e^+(x,\lambda) = \begin{pmatrix} o(1) \\ e^{i\lambda x} + o(1) \end{pmatrix}, \qquad \lambda \in \bar{\mathbf{C}}_+, \quad \lambda \to \infty \tag{2.5}$$

and

$$e^-(x,\lambda) = \begin{pmatrix} e^{-i\lambda x} + o(1) \\ o(1) \end{pmatrix}, \qquad \lambda \in \bar{\mathbf{C}}_+, \quad \lambda \to \infty. \tag{2.6}$$

From (2.5) and (2.6) we have

$$W\left\{e^{+},e^{-}\right\}=\lim_{x\to\infty}W\left\{e^{+}(x,\lambda),e^{-}(x,\lambda)\right\}=-1 \tag{2.7}$$

for $\lambda\in R$, where $W\left\{y^{(1)},y^{(2)}\right\}$ is the wronskian of the solutions of $y^{(1)}$ and $y^{(2)}$ which is defined as

$$W\left\{y^{(1)},y^{(2)}\right\}=y_1^{(1)}y_2^{(2)}-y_1^{(2)}y_2^{(1)},$$

here $y^{(k)}=\begin{pmatrix} y_1^{(k)} \\ y_2^{(k)} \end{pmatrix}$, $k=1,2$. Therefore $e^{+}(x,\lambda)$ and $e^{-}(x,\lambda)$ are the fundamental solutions of the system (1.1) for $\lambda\in\bar{\mathbf{C}}_{+}$. Now we will discuses the spectrum of $L(\lambda)$ defined as

$$\begin{aligned} iy_1'+q_1(x)y_2-\lambda y_1&=\varphi_1(x) \\ -iy_2'+q_2(x)y_1-\lambda y_2&=\varphi_2(x), \qquad x\in R_{+} \end{aligned} \tag{2.8}$$

and the condition

$$(a_1\lambda+b_1)y_2(0,\lambda)-(a_2\lambda+b_2)y_1(0,\lambda)=0 \tag{2.9}$$

where $\varphi(x)=\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$ is complex valued functions. We proved that $L(\lambda)$ has a finite number of eigenvalues and spectral singularities under the conditions

$$\sup_{x\in R_{+}}\left[\mid\varphi_k(x)\mid\exp(\varepsilon x^{\delta})\right]<\infty,\ k=1.2 \quad \varepsilon>0,\quad \frac{1}{2}<\delta<1 \tag{2.10}$$

by using analytic continuation method [7].

3 Eigen Values and Spectral Singularities of (2.8) and (2.9)

From (2.5), (2.6) and (2.7) it is not difficult to see that

$$\begin{aligned} y(x,\lambda)&=ie^{-}(x,\lambda)I^{+}(x,\lambda)-ie^{+}(x,\lambda)I^{-}(x,\lambda) \\ &\quad+a^{-}(\lambda)e^{+}(x,\lambda)-a^{+}(\lambda)e^{-}(x,\lambda), \end{aligned} \tag{3.1}$$

where $\lambda\in\bar{\mathbf{C}}_{+}$ and

$$\begin{aligned} I^{\pm}(x,\lambda)&=i\int_x^{\infty}\left(e_1^{\pm}(t,\lambda)\varphi_2(t)+e_2^{\pm}(t,\lambda)\varphi_1(t)\right)dt, \\ a^{\pm}(\lambda)&=iI^{\pm}(0,\lambda)+(a_1\lambda+b_1)e_2^{\pm}(0,\lambda)-(a_2\lambda+b_2)e_1^{\pm}(0,\lambda). \end{aligned}$$

Now, we have the following lemma:

Lemma 3.1. *If (2.1) and (2.10) hold then*

$$\sigma_d(L) = \left\{\lambda : \quad \lambda \in \mathbf{C}_+, \quad a^+(\lambda) = 0\right\} \tag{3.2}$$

where $\sigma_d(L)$ denotes the set of eigenvalues of the equations (2.8).

Proof. Let $\lambda_0 \in \mathbf{C}_+$. From (2.5) and (2.6) we get that $e^+(x, \lambda_0) \in L^2(R_+, C_2)$ and $e^-(x, \lambda_0) \notin L^2(R_+, C_2)$. Since

$$ie^-(x, \lambda_0) I^+ (x, \lambda) = o\left(\exp\left\{-\frac{\varepsilon}{2} x^\delta\right\}\right), \qquad x \to \infty,$$

and

$$ie^+(x, \lambda_0) I^- (x, \lambda) = o\left(\exp\left\{-\frac{\varepsilon}{2} x^\delta\right\}\right), \qquad x \to \infty,$$

it follows from (3.1) that $y(x, \lambda_0)$ belongs to $L^2(R_+, C_2)$ if and only if $a^+(\lambda) = 0$. Analogously from (3.1), we have

$$\sigma_{ss}(L) = \left\{\lambda : \quad \lambda \in R, \quad a^+(\lambda) = 0\right\} \tag{3.3}$$

where $\sigma_{ss}(L)$ denotes the set of spectral singularities of equations (2.8). It follows from (3.2) and (3.3) that in order to investigate the structure of the eigenvalues and the spectral singularities of equations (2.8), we need to discuss the structure of zeros of $a^+(\lambda)$ in $\bar{\mathbf{C}}_+$. In order to do so, we write the following:

$$M_1 = \left\{\lambda : \quad \lambda \in \mathbf{C}_+, \quad a^+(\lambda) = 0\right\}, \qquad M_2 = \left\{\lambda : \quad \lambda \in R, \quad a^+(\lambda) = 0\right\}.$$

Similarly from (3.2) and (3.3), we see that

$$\sigma_d(L) = \{\lambda : \quad \lambda \in M_1\}, \qquad \sigma_{ss}(L) = \{\lambda : \quad \lambda \in M_2\}. \tag{3.4}$$

□

Lemma 3.2. *If (2.1) and (2.10) hold, then the following occurs. The set M_1 is bounded and has a countable number of elements, and its limit points can lie in a bounded subinterval of the real axis. The set M_2 is compact.*

Proof. Using (2.2) and (2.10), we get that the function $a^+(\lambda)$ is analytic in $\mathbf{C}_+$, continuous in $\bar{\mathbf{C}}_+$, and

$$\begin{aligned} a^+(\lambda) = \quad & b_1 + \lambda\left\{a_1 + a_1 \int_0^\infty H_{22}(0,t) e^{i\lambda t} - a_2 \int_0^\infty H_{12}(0,t) e^{i\lambda t} dt\right\} \\ & + \int_0^\infty s^*(t) e^{i\lambda t} dt, \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} s^*(t) &= b_1 H_{22}(0,t) - b_2 H_{12}(0,t) \\ &\quad + i\left\{\varphi_1(t) + \varphi_2(t)\int_0^t H_{12}(t,s)ds + \varphi_1(t)\int_0^t H_{22}(t,s)ds\right\} \end{aligned}$$

Applying integration by parts we find

$$a^+(\lambda) = \lambda a_1 + b_1 + \int_0^\infty S^+(t)e^{i\lambda t}dt. \tag{3.6}$$

where

$$\begin{aligned} \lambda a_1 \int_0^\infty H_{22}(0,t)e^{i\lambda t}dt &= A + ia_1\int_0^\infty \frac{d}{dt}\left(H_{22}(0,t)\right)e^{i\lambda t}dt, \\ \lambda a_2 \int_0^\infty H_{12}(0,t)e^{i\lambda t}dt &= B + ia_2\int_0^\infty \frac{d}{dt}\left(H_{12}(0,t)\right)e^{i\lambda t}dt, \end{aligned}$$

and

$$\begin{aligned} S^+(t) &= \frac{d}{dt}\left(ia_1 H_{22}(0,t) - ia_2 H_{12}(0,t)\right) + b_1 H_{22}(0,t) \\ &\quad -b_2 H_{12}(0,t) + i\left\{\varphi_1(t) + \varphi_2(t)\int_0^t H_{12}(t,s)ds + \varphi_1(t)\int_0^t H_{22}(t,s)ds\right\} \end{aligned}$$

From (2.1), (2.4) and (2.10), it follows that

$$a^+(\lambda) = \lambda a_1 + O(1),\ \lambda \in \bar{\mathbf{C}}_+,\ |\lambda| \to \infty. \tag{3.7}$$

Equation (3.7) shows the boundedness of set M_1 and M_2. From the analyticity of the function $a^+(\lambda)$ in C_+ we obtain that M_1 has the most countable number of elements and its limit points can lie only in a bounded subinterval of the real axis. Using the boundary value uniqueness theorem of analytic functions, we obtain that the set M_2 is closed and $\mu(M_2) = 0$, where $\mu(M_2)$ denotes the linear Lebesque measure of M_2 [3]. □

From (2.1) and Lemma 3.2, we get the following theorem.

Theorem 3.3. *Under conditions (2.1) and (2.10), we have the following.*

(i) The set of eigenvalues of the equations (2.8), (2.9) is bounded, no more than countable and its limit points can lie only in a bounded subinterval of the positive semiaxis.

(ii) The set of spectral singularities of the equations (2.8), (2.9) is bounded and its linear Lebesque measure is zero.

Now, we recall the following.

Definition 1. *The multiplicity of zero of the function $a^+(\lambda)$ in $\bar{C}_+$ is defined as the multiplicity of the corresponding eigenvalue or spectral singularity of the equation (2.8), (2.9).*

Theorem 3.4. *If (2.10) holds and*

$$\sup_{x \in R_+} [\mid q_k(x) \mid \exp(\varepsilon x)] < \infty \quad k = 1.2 \quad \varepsilon > 0, \tag{3.8}$$

then the equation (2.8), (2.9) have a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. From (3.8) we find that

$$|H_{kj}(x,t)|, \left|\frac{d}{dt}H_{kj}(x,t)\right| \leq C\exp(-\frac{\varepsilon}{2}(x+t)), \quad k,j = 1,2 \tag{3.9}$$

where C>0 is a constant . (3.6),(3.9) imply that

$$\mid S^+(t) \mid \leq C\exp(-\frac{\varepsilon}{2}t), \tag{3.10}$$

It follows from (3.6) and(3.10) that the function $a^+(\lambda)$ has an analytic continuation from the real axis to the half plane $Im\lambda - (\varepsilon/2)$. So the limit points of the sets M_1 and M_2 can not lie in R, i.e., the bounded sets M_1 and M_2 have no limit points (see Lemma 3.2). Therefore, we have the finiteness of the zeros of $a^+(\lambda)$ in $\bar{C}_+$. Moreover, all zeros of $a^+(\lambda)$ in $\bar{C}_+$ have finite multiplicity. Using (3.4), we obtain that the equations (2.8), (2.9) have a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity. □

It is seen that conditions (2.10) and (3.8) guarantee the analytic continuation of the function $a^+(\lambda)$ from real axis to lower half-plane. So the finiteness of eigenvalues and spectral singularities of the equations (2.8), (2.9) are obtained as a result of this analytic continuation. Now let us suppose that

$$\sup_{x \in R_+} [\mid q_k(x) \mid \exp(\varepsilon x^\delta)] < \infty, \; k = 1.2 \quad \varepsilon > 0, \quad \frac{1}{2} < \delta < 1, \tag{3.11}$$

which is weaker than (3.8). It is evident that under the conditions (2.10) and (3.11), the function $a^+(\lambda)$ is analytic in $\bar{C}_+$ and infinitely differentiable on the real axis. But $a^+(\lambda)$ does not have an analytic continuation from the real axis to lower half-plane. Therefore, under the condition (2.10), the finiteness of eigenvalues and spectral singularities of the equations (2.8), (2.9) can not be proved by the same technique used in theorem 3.2. Let us denote the sets of all limit points of $M_1$by M_3 and the set of all zeros of $a^+(\lambda)$ with infinite multiplicity in $\bar{C}_+$ by M_4. It follows from the boundary uniqueness theorem of analytic functions that

$$M_1 \cap M_4 = \emptyset, \qquad M_3 \subset M_2, \qquad M_4 \subset M_2,$$

and

$$\mu(M_3) = \mu(M_4) = 0.$$

Using the continuity of all derivatives of $a^+(\lambda)$ on the real axis, we have

$$M_3 \subset M_4. \tag{3.12}$$

To prove the next result we will use the following uniqueness theorem for the analytic functions on the upper half-plane.

Theorem 3.5. *(see* [10].*) Let us assume that the function g is analytic in $\bar{C}_+$, all of its derivatives are continuous up to the real axis and there exists $T > 0$ such that*

$$\left|g^{(n)}(z)\right| \leq C_n, \qquad n = 0, 1, 2, ..., \lambda \in C_+, \quad |z| < 2T, \tag{3.13}$$

and

$$\left|\int_{-\infty}^{-T} \frac{\ln|g(x)|}{1+x^2} dx\right| < \infty, \qquad \left|\int_{T}^{\infty} \frac{\ln|g(x)|}{1+x^2} dx\right| < \infty. \tag{3.14}$$

If the set Q with linear Lebesgue measure zero, is the set of all zeros of the function g with infinite multiplicity and if

$$\int_0^a \ln F(s) d\mu(Q_s) = -\infty \tag{3.15}$$

then $g(z) \equiv 0$, where $F(s) = \inf_n \left(\frac{C_n s^n}{n!}\right)$, $\quad n = 0, 1, ...,$ $\mu(Q_s)$ is the linear Lebesgue measure of s-neighborhood of Q and a is an arbitrary positive constant.

Lemma 3.6. *If (2.10) and (3.11) hold, then $M_4 = \emptyset$.*

Proof. It follows from (3.8)-(3.10) that the function $a^+(\lambda)$ is analytic in C_+ and all of its derivatives are continuous up to real axis. Moreover, by Lemma 3.2. for sufficiently large $T > 0$, we have

$$\left|\int_{-\infty}^{-T} \frac{\ln|a^+(\lambda)|}{1+x^2} dx\right| < \infty, \qquad \left|\int_{T}^{\infty} \frac{\ln|a^+(\lambda)|}{1+x^2} dx\right| < \infty. \tag{3.16}$$

From (3.6), we obtain

$$\left|a^{+}(\lambda)-\lambda a_1\right|<\infty, \qquad \lambda\in\bar{\mathbf{C}}_{+}, \tag{3.17}$$

and

$$\left|\frac{d^n}{d\lambda^n}a^{+}(\lambda)\right|\le D_n, \qquad n=1,2,..., \tag{3.18}$$

where

$$D_1=a_1+\int_0^\infty tS^{+}(t)dt, \qquad D_n=\int_0^\infty t^nS^{+}(t)dt, \qquad n=2,3,... \tag{3.19}$$

Using (2.4)

$$|H_{kj}(x,t)|\le c\exp\left[-\frac{\varepsilon}{2}\left(\frac{x+t}{2}\right)^{\delta}\right], \quad k,j=1,2, \qquad \varepsilon>0, \quad \frac{1}{2}<\delta<1,$$

and consequently,

$$S^{+}(t)\le c\exp\left[-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^{\delta}\right], \qquad \varepsilon>0, \quad \frac{1}{2}<\delta<1, \tag{3.20}$$

where C>0 is a constant.From (3.17)-(3.20), we obtain

$$\left|\frac{d^n}{d\lambda^n}a^{+}(\lambda)\right|\le K_n, \quad n=1,2,...,\lambda, \quad |\lambda|<2T, \tag{3.21}$$

where

$$K_n=C\int_0^\infty t^n\exp\left[-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^{\delta}\right]dt. \tag{3.22}$$

It is easy to see from (3.16) and (3.21) that $a^{+}(\lambda)$ satisfies (3.14) and (3.15). Since the function $a^{+}(\lambda)$ is not equal to zero identically, then by Theorem 3.3., M_4 satisfies

$$\int_0^a \ln F(s)d\mu(\,M_{4,s})>-\infty, \tag{3.23}$$

where $F(s)=\inf_n\left(\frac{K_ns^n}{n!}\right)$, $\mu(\,M_{4,s})$ is the linear Lebesque measure of s-neighborhood of M_4 and the constant K_n is defined by (3.22). Now we will obtain the following estimates for K_n:

$$K_n=C\int_0^\infty t^n\exp\left[-\frac{\varepsilon}{2}\left(\frac{t}{2}\right)^{\delta}\right]dt\le Bb^nn!n^{n(1-\delta)/\delta}, \tag{3.24}$$

where B and b are constants depending on C, ϵ and δ. Substituting (3.23) in the definition of $F(s)$, we arrive at

$$F(s) = \inf_n \left(\frac{K_n s^n}{n!} \right) \leq B \exp \left\{ -\frac{1-\delta}{\delta} e^{-1/(1-\delta)} b^{-\delta/(1-\delta)} s^{-\delta/(1-\delta)} \right\}.$$

Now by (3.23), we obtain

$$\int_0^a s^{-\delta/(1-\delta)} d\mu(M_{4,s}) < \infty. \tag{3.25}$$

Since $\delta/ (1-\delta) \geq 1$, (3.25) holds for arbitrary s if and only if $\mu(M_{4,s}) = 0$ or $M_4 = \emptyset$. □

Theorem 3.7. *Under the conditions (2.10) and (3.11), The equations (2.8) and (2.9) have finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof. To be able to prove that, we have to show that the function $a^+(\lambda)$ has a finite number of zeros with a finite multiplicities in $\bar{C}_+$. It follows from (3.12) and Lemma 3.3 that $M_3 = \emptyset$. So the bounded set M_1 has no limit points,i.e., The function $a^+(\lambda)$ has finite number of zeros in $\bar{C}_+$. Since $M_4 = \emptyset$, these zeros are of finite multiplicity. □

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