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Construction of the smallest common coarser of two and three set partitions

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Abstract

This paper is inspired by a text of the book [7] ("Úvod do algebry" in Czech, "Introduction to Algebra" in English) of the authors Ladislav Kosmák and Radovan Potůček. They followed the great work of Professor Otakar Borůvka in the field of the partition theory, groupoids and groups and gave them in the context to contemporary modern algebra. Academian Borůvka have deduced and proved many results concerning the partition theory in his publications.

His first works [1] and [2] were published during World War II and his monographs [3] and [4] were released in the post-war years.

In this paper we deal with a construction of the smallest common coarser of two set partitions associated with equivalence relations, we give a special relation used in the construction and an illustration of blocks of this coarser.

Remark 1. Algebra of binary relations was worked up by Jacques Riguet in 1948 (see [8], the translation into Russian has its origin from 1963). Professor Borůvka was rightfully proud himself on his theory of partitions and an appliance of the algebra of relations has never used. From twenty-seven chapters of the book [4] is six devoted to the general theory of partitions and in the chapter 14 and $21-24$ are paragraphs dealing with partitions in the theory of grupoids and groups.

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In recent decades there has been a rapid development of the theory and applications of algebraic structures but also the algebraic hyperstructures. In this context, we may refer to interesting articles [5] of Cristine Flaut and [6] of Jan Chvalina, Šárka Hošková-Mayerová, and Dehghan Nezhad.

First, let us briefly recall some basic and well-known terms, notions and facts, without proofs, concerning partitions and equivalence relations.

Definition 1. A *partition* of a set M is a collection $\mathcal{S}$ of nonempty subsets of M, called blocks, parts or cells of the partition, such that:

1. All sets in $\mathcal{S}$ are pairwise disjoint, i.e. $\forall\, S_1, S_2 \in \mathcal{S}\colon S_1 \cap S_2 = \emptyset$ when $S_1 \neq S_2$.

2. The union of all the sets forms the whole set M, i.e. $\displaystyle\bigcup_{S_i \in \mathcal{S}} S_i = M$.

Definition 2. Let $\mathcal{R}, \mathcal{S}$ be partitions of a set M and let for every $X \in \mathcal{R}$ there exists such $Y \in \mathcal{S}$ that $X \subset Y$. Then the partition $\mathcal{S}$ is said to be a *coarser partition* than $\mathcal{R}$, the partition $\mathcal{R}$ is said to be a *finer partition* than $\mathcal{S}$, and we write $\mathcal{R} \sqsubset \mathcal{S}$.
If it holds $\mathcal{R} \sqsubset \mathcal{S}$, we also say that the partition $\mathcal{R}$ is a *refinement* of the partition $\mathcal{S}$.

Definition 3. A *meet* of two partitions $\mathcal{R}, \mathcal{S}$ of a set M, denoted by $\mathcal{R} \sqcap \mathcal{S}$, is a set of all intersections $X \cap Y$ where $X \in \mathcal{R}$, $Y \in \mathcal{S}$.

Definition 4. A *power set* of a set M, denoted $\mathcal{P}(M)$, is the set of all subsets of M, i.e. for any sets M and X it holds: $X \in \mathcal{P}(M)$ if and only if $X \subseteq M$.

Theorem 1. *For any parts $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of a set $\mathcal{P}(M)$ it holds:*

1. *$\mathcal{A} \sqcap \mathcal{B} \sqsubset \mathcal{A}$, $\mathcal{A} \sqcap \mathcal{B} \sqsubset \mathcal{B}$.*

2. *If $\mathcal{C} \sqsubset \mathcal{A}$ and $\mathcal{C} \sqsubset \mathcal{B}$, then $\mathcal{C} \sqsubset \mathcal{A} \sqcap \mathcal{B}$.*

Theorem 2. *If $\mathcal{R}, \mathcal{S}$ are partitions of a set M such that $\mathcal{R} \sqsubset \mathcal{S}$ and $\mathcal{S} \sqsubset \mathcal{R}$, then $\mathcal{R} = \mathcal{S}$.*

Definition 5. An *equivalence relation* on a set M is a relation R on M such that:

1. $(x, x) \in R$ for all $x \in M$, i.e the relation R is reflexive.

2. If $(x, y) \in R$, then $(y, x) \in R$, i.e. the relation R is symmetric.

3. If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$, i.e. the relation R is transitive.

An *equivalence class* of an element $x \in M$ is defined as the set $\{y \in M;\, (x, y) \in R\}$.

Remark 2. Denote an identity relation $\{(x, y) \in M \times M;\, x = y\}$ by E, an inverse relation $\{(y, x) \in M \times M;\, (x, y) \in R\}$ by R^{-}, and a composition of relations $R \circ R = \{(x, z) \in M \times M;\, \exists\, y \in M \colon (x, y) \in R \wedge (y, z) \in S\}$ by R^2, we can write reflexivity in the form $E \subset R$, symmetry by the equality $R = R^{-}$, and transitivity as $R^2 \subset R$.

Theorem 3. *Each partition $\mathcal{R}$ of a set M induces an associated equivalence relation R on M, and conversely each equivalence relation R on M induces an associated partition $\mathcal{R}$ of M, denoted by M/R, into equivalences classes. Thus there is a bijection from the set of all possible equivalence relations on M to the set of all partitions of M.*

Now, let us come to our main topic:

Theorem 4. *For equivalence relations R, S on a set M it holds $R \subset S$ if and only if $M/R \sqsubset M/S$.*

Proof. Let $R \subset S$ and let $a \in M$. Then for each $x \in M$, such that xRa, we have xSa, thus $R_a \subset S_a$, i.e. $M/R \sqsubset M/S$.

When contrariwise $M/R \sqsubset M/S$, we get $R_a \subset S_a$ for each $a \in M$, and thus from xRa it follows xSa. □

Theorem 5. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two partitions of a set M and let*

$$\mathcal{R} = (\mathcal{R}_1 \sqcap \mathcal{R}_2) \smallsetminus \{\emptyset\}.$$

Then $\mathcal{R}$ is a partition of M and it holds:

1. $\mathcal{R} \sqsubset \mathcal{R}_1$, $\mathcal{R} \sqsubset \mathcal{R}_2$.
2. *If $\mathcal{S}$ is a partition of M such that $\mathcal{S} \sqsubset \mathcal{R}_1$ and $\mathcal{S} \sqsubset \mathcal{R}_2$, then $\mathcal{S} \sqsubset \mathcal{R}$.*

Proof. If $A, B \in \mathcal{R}$, then there exist sets $A_1, B_1 \in \mathcal{R}_1$, $A_2, B_2 \in \mathcal{R}_2$, such that $A = A_1 \cap A_2$, $B = B_1 \cap B_2$. If $A \neq B$, then there exists $k \in \{1, 2\}$, such that $A_k \neq B_k$, so $A_k \cap B_k = \emptyset$, and thus $A \cap B = \emptyset$. By Definition 1, for any $x \in M$ there exists a block X_1 of the partition $\mathcal{R}_1$ and a block X_2 of the partition $\mathcal{R}_2$, both the blocks containing the element x, so $x \in X_1 \cap X_2 \in \mathcal{R}$. The statements 1. and 2. follow from Theorem 1. □

Definition 6. The partition $\mathcal{R}$ from Theorem 5 is called the *greatest common refinement* of the partitions $\mathcal{R}_1, \mathcal{R}_2$ and is denoted by $\mathcal{R}_1 \wedge \mathcal{R}_2$.

Theorem 6. *Let $\mathcal{R}_1, \mathcal{R}_2$ be two partitions of a set M and let $\mathcal{S}_0$ is a meet of a system Υ of all partitions of M, that all are coarser than partitions $\mathcal{R}_1$ and $\mathcal{R}_2$. Then*

$$\mathcal{S} = \mathcal{S}_0 \smallsetminus \{\emptyset\}$$

is a partition of M and it holds:

1. $\mathcal{R}_1 \sqsubset \mathcal{S}$, $\mathcal{R}_2 \sqsubset \mathcal{S}$.

2. *If* $\mathcal{T}$ *is any partition of* M *that is coarser than both the partitions* $\mathcal{R}_1, \mathcal{R}_2$, *then* $\mathcal{S} \sqsubset \mathcal{T}$.

Proof. As $\mathcal{R}_1 \sqsubset \mathcal{R}$, $\mathcal{R}_2 \sqsubset \mathcal{R}$ for each $\mathcal{R} \in \Upsilon$, it holds, by generalized Theorem 5, $\mathcal{R}_1 \sqsubset \mathcal{S}$, $\mathcal{R}_2 \sqsubset \mathcal{S}$.

If $\mathcal{T} \in \Upsilon$, then $\mathcal{T}$ is coarser then the meet $\mathcal{S}_0$ by its definition, so we get $\mathcal{S} \sqsubset \mathcal{T}$. □

Remark 3. The partition $\mathcal{S}$ from Theorem 6 is thus the *smallest common coarser* of the partitions $\mathcal{R}_1, \mathcal{R}_2$. A construction of an equivalence relation, which determine this partition, is much more difficult than in a case of the greatest common refinement. We give the construction of the smallest common coarser in the following theorems.

Theorem 7. *Let* R, S *be equivalence relations on a set* M *and let a relation* T_R *on* M *is defined this way:* $(x, y) \in T_R$ *if and only if there exists such a natural number* k *that*

$$(x, y) \in (RSR)^k.$$

Then T_R *is the equivalence relation.*

Proof. Since the relation R, S are reflexive, it holds

$$E \subset (RSR)^n$$

for all natural number n, thus $E \subset T_R$.

As the relation R, S are symmetric, we have

$$(RSR)^- = R^- S^- R^- = RSR,$$

and hence $[(RSR)^n]^- = (RSR)^n$ for all natural numbers n, so $T_R^- = T_R$.

A transitivity of the relations R, S implies that for any natural numbers k, l it holds

$$(RSR)^k (RSR)^l = (RSR)^{k+l},$$

so the relation T_R is transitive. □

Remark 4. From a definition of the relation T_R it follows that

$$T_R = \bigcup_{k=0}^{\infty} (RSR)^k, \quad \text{where} \quad (RSR)^0 = E.$$

Theorem 8. *The partition $\mathcal{T}$ determined by the equivalence relation T_R from Theorem 7 is a common coarser of the partitions $\mathcal{R}, \mathcal{S}$ associated with the equivalence relations R, S.*

Proof. The reflexivity of the relations R, S implies

$$R = RE \subset RS = RSE \subset RSR \subset (RSR)^n \subset T_R,$$
$$S = ESE \subset RSR \subset (RSR)^n \subset T_R$$

for all natural numbers n. Hence $R \subset T_R$ a $S \subset T_R$, and by Theorem 2 we have

$$\mathcal{R} \sqsubset \mathcal{T}, \quad \mathcal{S} \sqsubset \mathcal{T}.$$

□

Theorem 9. *At a notation used in Theorem 8, let $\mathcal{V}$ is an arbitrary common coarser of the partitions $\mathcal{R}, \mathcal{S}$. Then*

$$\mathcal{T} \sqsubset \mathcal{V},$$

hence $\mathcal{T}$ is the smallest common coarser of the partitions $\mathcal{R}, \mathcal{S}$.

Remark 5. For the partition $\mathcal{T}$ from Theorem 9 we shall use a notation $\mathcal{R} \vee \mathcal{S}$.

Proof. If V is an equivalence relation associated with the partition $\mathcal{V}$, then it holds $R \subset V$, $S \subset V$, thus for all natural numbers k we have

$$(RSR)^k \subset V^{3k}.$$

A transitivity of the relation relace V implies that

$$V \supset V^2 \supset V^3 \supset \cdots,$$

so

$$T_R = \bigcup_{k=0}^{\infty} (RSR)^k \subset \bigcup_{k=0}^{\infty} V^k = V.$$

Hence $T_R \subset V$, and by Theorem 2 we get $\mathcal{T} \sqsubset \mathcal{V}$. This proves that

$$\mathcal{T} = \mathcal{R} \vee \mathcal{S}.$$

□

Theorem 10. *Let R, S be equivalence relations on a set M and let a relation T_S on M is defined this way: $(x, y) \in T_S$ if and only if there exists such a natural number k that*

$$(x, y) \in (SRS)^k.$$

Then $T_S = T_R$.

Proof. The inclusions

$$RSR \subset S(RSR)S = (SRS)(SRS) \subset SRS,$$
$$SRS \subset R(SRS)R = (RSR)(RSR) \subset RSR$$

imply the equality

$$RSR = SRS,$$

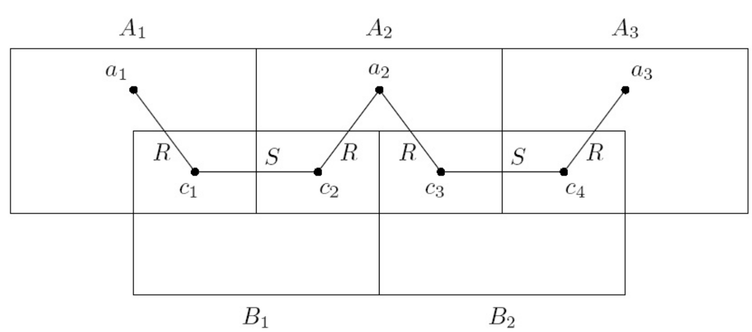
and hence

$$T_S = \bigcup_{k=0}^{\infty} (SRS)^k = \bigcup_{k=0}^{\infty} (RSR)^k = T_R. \qquad \square$$

Corollary 1. Both equivalence relations T_R, T_S determine the same partition $\mathcal{R} \vee \mathcal{S}$.

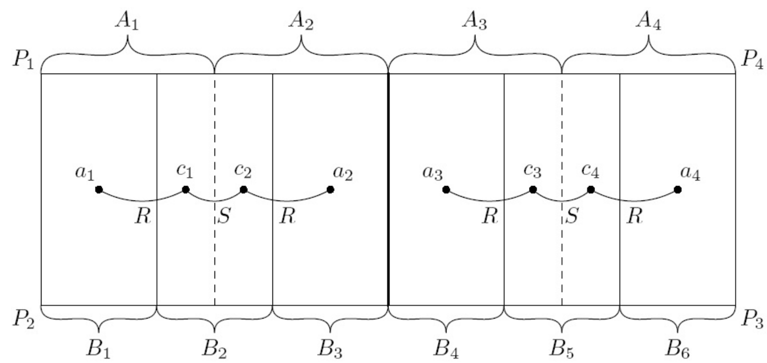
Remark 6. Now, we show a representation of the relation RSR, where A_1, A_2, A_3 are blocks of the partition $\mathcal{R}$ and B_1, B_2 are blocks of the partition $\mathcal{S}$, and a construction of the smallest common coarser of two partitions. In the picture 1.1 below is schematically presented a relation

$$a_1 R c_1 S c_2 R a_2 R c_3 S c_4 R a_3, \quad \text{i.e.} \quad a_1 RSR a_2 RSR a_3, \quad \text{i.e.} \quad a_1 (RSR)^2 a_3.$$



Pic. 1.1

In the picture 1.2 a set M is the rectangle $P_1P_2P_3P_4$, $A_1, \ldots, A_4$ are blocks of the partition $\mathcal{R}$, $B_1, \ldots, B_6$ are blocks of the partition $\mathcal{S}$, and $A_1 \cup A_2$, $A_3 \cup A_4$ are blocks of the smallest common coarser $\mathcal{R} \vee \mathcal{S}$. Two blocks of the smallest common coarser are also presented:



Pic. 1.2

Theorem 11. *Let R, S, T be equivalence relations on a set M and let a relation V_1 on M is defined this way: $(x, y) \in V_1$ if and only if there exists such a natural number k that*

$$(x, y) \in (RSTSR)^k.$$

Then V_1 is the equivalence relation.

Proof. Since the relations R, S, T are reflexive, it holds

$$E \subset (RSTSR)^n$$

for all natural number n, thus $E \subset V_1$.

As the relations R, S, T are symmetric, we have

$$(RSTSR)^- = R^- S^- T^- S^- R^- = RSTSR,$$

and hence

$$[(RSTSR)^n]^- = (RSTSR)^n.$$

If there exist such natural numbers k, l, that

$$(x, y) \in (RSTSR)^k,$$

$$(y, z) \in (RSTSR)^l,$$

then

$$(x, y) \in (RSTSR)^k (RSTSR)^l = (RSTSR)^{k+l},$$

so the relation V_1 is transitive. $\square$

Theorem 12. *Let R, S, T be equivalence relations on a set M and let a relation V_2 on M is defined this way: $(x, y) \in V_2$ if and only if there exists such a natural number k that*

$$(x, y) \in (TSRST)^k.$$

Then V_2 is the equivalence relation.

Proof. Is similar to the proof of Theorem 11, where the relations R and T are mutually replaced. □

Theorem 13. *At a notation used in Theorems* 11 *and* 12 *it holds*

$$V_1 = V_2.$$

Proof. The inclusions

$$TSRST \subset RS(TSRST)SR \subset (RSTSR)(RSTSR) \subset RSTSR,$$
$$RSTSR \subset TS(RSTSR)ST \subset (TSRST)(TSRST) \subset TSRST,$$

imply the equality

$$RSTSR = TSRST,$$

and hence

$$V_1 = \bigcup_{k=0}^{\infty} (RSTSR)^k = \bigcup_{k=0}^{\infty} (TSRST)^k = V_2.$$ □

Theorem 14. *The partition $\mathcal{V}$ determined by the equivalence relations V_1, V_2 is the smallest common coarser of the partitions $\mathcal{R}, \mathcal{S}, \mathcal{T}$ associated with the equivalence relations R, S, T.*

Proof. We have

$$R \subset RS \subset RST \subset RSTS \subset RSTSR \subset V_1$$

and similarly $S \subset V_1$, $T \subset V_1$, so the partition $\mathcal{V}$ is the common coarser of the partitions $\mathcal{R}, \mathcal{S}, \mathcal{T}$.

If $\mathcal{W}$ is an arbitrary common coarser of the partitions $\mathcal{R}, \mathcal{S}, \mathcal{T}$ and if W is an equivalence relation which determines this partition, then they hold the inclusions

$$R \subset W, \quad S \subset W, \quad T \subset W,$$

and hence for every natural number n we get

$$(RSTSR)^n \subset W^{5n}.$$

Since the relation W is transitive, we have

$$W \supset W^2 \supset W^3 \supset \cdots,$$

and so

$$V = \bigcup_{k=1}^{\infty} (RSTSR)^k \subset W \cup W^2 \cup W^3 \cup \cdots = W.$$

By Theorem 4, we have proved that $\mathcal{V}$ is the smallest common coarser of the partitions $\mathcal{R}, \mathcal{S}, \mathcal{T}$. $\square$

Remark 7. Denote by $\mathcal{R} \vee \mathcal{S}$ the smallest common coarser of the partitions $\mathcal{R}, \mathcal{S}$, we obtain the formulas

1. $\mathcal{R} \vee \mathcal{S} = \mathcal{S} \vee \mathcal{R}$,
2. $\mathcal{R} \vee (\mathcal{S} \vee \mathcal{T}) = (\mathcal{R} \vee \mathcal{S}) \vee \mathcal{T}$.

as the results of Theorems 10 and 13.

At the proof of the proposition 2., it was useful to apply the equality

$$(\mathcal{R} \vee \mathcal{S}) \vee \mathcal{T} = \mathcal{T} \vee (\mathcal{R} \vee \mathcal{S}).$$

Theorem 15. *For arbitrary partitions $\mathcal{R}, \mathcal{S}$ of a set M they hold the equalities*

$$\mathcal{R} \vee (\mathcal{R} \wedge \mathcal{S}) = \mathcal{R},$$
$$\mathcal{R} \wedge (\mathcal{R} \vee \mathcal{S}) = \mathcal{R}.$$

Proof. Both propositions of the theorem follow from the relations

$$\mathcal{R} \wedge \mathcal{S} \sqsubset \mathcal{R},$$
$$\mathcal{R} \sqsubset \mathcal{R} \vee \mathcal{S}.$$

$\square$

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