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About k-perfect numbers

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Abstract

ABSTRACT. In this paper we present some results about k-perfect numbers, and generalize two inequalities due to M. Perisastri (see [6]).

1 Introduction

Definition. A positive integer n is k-perfect if $\sigma(n) = kn$, when $k > 1$, $k \in Q$. The special case $k = 2$ corresponds to perfect numbers, which are intimately connected with Mersenne primes. We have the following smallest k-perfect numbers. For $k = 2$ $(6, 28, 496, 8128, ...)$, for $k = 3$ $(120, 672, 523776, 459818240, ...)$, for $k = 4$ $(30240, 32760, 2178540, ...)$, for $k = 5$ $(14182439040, 31998395520, ...)$, for $k = 6$ $(154345556085770649600, ...)$.

For a given prime number p, if n is p-perfect and p does not divide n, then pn id $(p+1)-$ perfect. This imples that an integer n is a $3-$ perfect number divisible by 2 but not by 4, if and only if $\frac{n}{2}$ is an odd perfect number, of which none are known. If $3n$ is $4k-$ perfect and 3 does not divide n, then n is $3k-$perfect.

A $k-$perfect number is a positive integer n such that its harmonic sum of divisors is k.

For the perfect numbers we have the followings: $28 = 1^3 + 3^3$, $496 = 1^3 + 3^3 + 5^3 + 7^3$, $8128 = 1^3 + 3^3 + 5^3 + 7^3 + 9^3 + 11^3 + 13^3 + 15^3$ etc. We posted the following conjecture:

Conjecture. (Bencze, M., 1978) If n is k-perfect, then exist odd positive integers u_i $(i = 1, 2, ..., r)$ such that

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$$n = \sum_{i=1}^{r} u_i^{k+1}$$

MAIN RESULTS

Theorem 1. If $f : R \to R$ is convex and increasing, $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ written in cannonical form is k-perfect, then:

$$\sum_{i=1}^{n} f\left(\frac{1}{p_i}\right) \geq \begin{cases} nf\left(\sqrt[n]{\frac{3}{2}} - 1\right) & \text{if} \quad N \quad \text{is even} \\ nf\left(\sqrt[3n]{k^2} - 1\right) & \text{if} \quad N \quad \text{is odd} \end{cases}$$

Proof. If N is even then it follows

$$\prod_{i=1}^{n} \frac{p_i + 1}{p_i} > \frac{3}{2}$$

For $x \geq 3$ holds $\frac{x+1}{x} \geq \sqrt[3]{\left(\frac{x}{x-1}\right)^2}$ (see [9]), therefore if N is odd then yields

$$\prod_{i=1}^{n} \frac{p_i + 1}{p_i} > \sqrt[3]{\prod_{i=1}^{n} \left(\frac{p_i}{p_i - 1}\right)^2} > \sqrt[3]{k^2}$$

because

$$\prod_{i=1}^{n} \frac{p_i}{p_i - 1} = k \prod_{i=1}^{n} \frac{p_i^{\alpha_i + 1}}{p_i^{\alpha_i + 1} - 1} > k$$

Using the AM-GM inequality we obtain:

$$\prod_{i=1}^{n} \frac{p_i + 1}{p_i} \leq \left(\frac{1}{n} \sum_{i=1}^{n} \frac{p_i + 1}{p_i}\right)^n = \left(1 + \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}\right)^n$$

Finally

$$\sum_{i=1}^{n} \frac{1}{p_i} > \begin{cases} n\left(\sqrt[n]{\frac{3}{2}} - 1\right) & \text{if} \quad N \quad \text{is even} \\ n\left(\sqrt[3n]{k^2} - 1\right) & \text{if} \quad N \quad \text{is odd} \end{cases}$$

Because f is convex and increasing from Jensen's inequality we get

$$\sum_{i=1}^{n} f\left(\frac{1}{p_i}\right) \geq nf\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}\right) \geq \begin{cases} nf\left(\sqrt[n]{\frac{3}{2}}-1\right) & \text{if} \quad N \quad \text{is even} \\ nf\left(\sqrt[3n]{k^2}-1\right) & \text{if} \quad N \quad \text{is odd} \end{cases} \tag{1}$$

Theorem 2. If $g : R \to R$ is convex and increasing, $N = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$ written in cannonical form is k-perfect, then:

$$\sum_{i=1}^{n} g\left(\frac{1}{p_i}\right) \leq \begin{cases} ng\left(1-\sqrt[n]{\frac{6}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is even} \\ ng\left(1-\sqrt[n]{\frac{8}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is odd} \end{cases}$$

Proof. We have the following:

$$\prod_{i=1}^{n}\frac{p_i}{p_i-1} = \prod_{i=1}^{n}\frac{p_i^{\alpha_i+1}-1}{(p_i-1)\,p_i^{\alpha_i}}\prod_{i=1}^{n}\frac{p_i^{\alpha_i+1}}{p_i^{\alpha_i+1}-1} = k\prod_{i=1}^{n}\frac{1}{1-\frac{1}{p_i^{\alpha_i+1}}} = k\prod_{i=1}^{n}\left(\sum_{j=0}^{\infty}\left(\frac{1}{p_i}\right)^j\right) \leq$$

$$\leq k\prod_{i=1}^{n}\left(\sum_{j=0}^{\infty}\frac{1}{p_i^{2j}}\right) < \begin{cases} k\sum_{n=1}^{\infty}\frac{1}{n^2} & \text{if} \quad N \quad \text{is even} \\ k\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} & \text{if} \quad N \quad \text{is odd} \end{cases} = \begin{cases} \frac{k\pi^2}{6} & \text{if} \quad N \quad \text{is even} \\ \frac{k\pi^2}{8} & \text{if} \quad N \quad \text{is odd} \end{cases}$$

From AM-GM inequality yields

$$\prod_{i=1}^{n}\frac{p_i}{p_i-1} \geq \left(\frac{n}{\sum_{i=1}^{n}\frac{p_i-1}{p_i}}\right)^n = \left(\frac{n}{n-\sum_{i=1}^{n}\frac{1}{p_i}}\right)^n$$

therefore

$$\sum_{i=1}^{n}\frac{1}{p_i} < \begin{cases} n\left(1-\sqrt[n]{\frac{6}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is even} \\ n\left(1-\sqrt[n]{\frac{8}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is odd} \end{cases}$$

According to Jensen's inequality yields

$$\sum_{i=1}^{n} g\left(\frac{1}{p_i}\right) \leq ng\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}\right) \leq \begin{cases} ng\left(1-\sqrt[n]{\frac{6}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is even} \\ ng\left(1-\sqrt[n]{\frac{8}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is odd} \end{cases}$$

Corolloary 1. If $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ written in cannonical form is k-perfect then:

$$\begin{cases} n\left(\sqrt[n]{\frac{3}{2}} - 1\right) & \text{if} \quad N \quad \text{is even} \\ n\left(\sqrt[3n]{k^2} - 1\right) & \text{if} \quad N \quad \text{is odd} \end{cases} < \sum_{i=1}^{n} \frac{1}{p_i} < \begin{cases} n\left(1 - \sqrt[n]{\frac{6}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is even} \\ n\left(1 - \sqrt[n]{\frac{8}{k\pi^2}}\right) & \text{if} \quad N \quad \text{is odd} \end{cases}$$

Theorem 3. If $x, t > 0$ then

$$(x+1)\, t^{\frac{1}{x+1}} - x t^{\frac{1}{x}} \leq 1$$

Proof. For $t = 1$ we have the equality. Let $0 < t < 1$. Since the function $u(x) = x t^{\frac{1}{x}}$ is continuous and differentiable we can apply the Lagrange's theorem and we obtain

$$\frac{(x+1)\, t^{\frac{1}{x+1}} - x t^{\frac{1}{x}}}{(x+1) - x} = \frac{u(x+1) - u(x)}{(x+1) - x} = u'(z)$$

when $x < z < x + 1$ hence we have the inequality

$$t^{\frac{1}{z}} \left(1 - \frac{1}{z} \ln t\right) < 1 \text{ or } 1 - \frac{1}{z} \ln t < t^{-\frac{1}{z}}.$$

Developing $t^{-\frac{1}{z}}$ into McLauren's series it results

$$1 - \frac{1}{z} \ln t < 1 - \frac{1}{1!z} \ln t + \frac{1}{2!z^2} \ln^2 t - \frac{1}{3!z^3} \ln^3 t + ...$$

or

$$\sum_{r=2}^{\infty} \frac{(-1)^r \ln^r t}{r! z^r} > 0 \text{ or } \sum_{r=2}^{\infty} \frac{\ln^r \frac{1}{t}}{r! z^t} > 0$$

that is obvious because $\ln \frac{1}{t} > 0$ due to $\frac{1}{t} > 1$. Let be $t > 1$. Then is enough to show that the function $V(x) = x\left(t^{\frac{1}{x}} - 1\right)$ is decreasing.

Differentiable V we get

$$V'(x) = t^{\frac{1}{x}} - t^{\frac{1}{x}} \cdot \frac{1}{x} \ln t - 1 = -\sum_{r=2}^{\infty} \frac{\ln^r t}{x^r (r-1)!} \left(1 - \frac{1}{r}\right) < 0$$

Since V is decreasing and we may say that $V(x+1) < V(x)$ hence and from it follows the inequality of the ennunciation.

Corollary 2. If $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ is a k-perfect number written in canonical form, then:

$$\begin{cases} \ln\frac{3}{2} & \text{if} \quad N \quad \text{is even} \\ \frac{2}{3}\ln k & \text{if} \quad N \quad \text{is odd} \end{cases} < \sum_{i=1}^{n}\frac{1}{p_i} < \begin{cases} \ln\frac{k\pi^2}{6} & \text{if} \quad N \quad \text{is even} \\ \ln\frac{k\pi^2}{8} & \text{if} \quad N \quad \text{is odd} \end{cases}$$

Proof. Using the Theorem 3 it is proved that the series

$$\left(n\left(\sqrt[n]{\frac{3}{2}}-1\right)\right)_{n\in N^*} \quad \text{and} \quad \left(n\left(\sqrt[3n]{k^2}-1\right)\right)_{n\in N^*}$$

are decreasing, and the series

$$\left(n\left(1-\sqrt[n]{\frac{6}{k\pi^2}}\right)\right)_{n\in N^*} \quad \text{and} \quad \left(n\left(1-\sqrt[n]{\frac{8}{k\pi^2}}\right)\right)_{n\in N^*}$$

are increasing. It means that the minimum and maximum are reached only then $n \to \infty$.

Since $n \to \infty$ we have $0 \cdot \infty$. That is why L'Hospital rule and so we find the results of the enunciation.

Remark 1. *For $k = 2$ we reobtain the M.Perisastri's inequality*

$$\sum_{i=1}^{n}\frac{1}{p_i} < 2\ln\frac{\pi}{2}$$

(see [6]).

Corollary 3. Let $N = p_1^{\alpha_1}p_2^{\alpha_2}...p_n^{\alpha_n}$ be a $k-$perfect number written in cannonical form and $P_{\max} = \{p_1, p_2, ..., p_n\}$ and $P_{\min} = \min\{p_1, p_2, ..., p_n\}$, then

$$P_{\min} < \begin{cases} \frac{1}{\sqrt[n]{\frac{3}{2}}-1} & \text{if} \quad N \quad \text{is even} \\ \frac{1}{\sqrt[3n]{k^2}-1} & \text{if} \quad N \quad \text{is odd} \end{cases}$$

and

$$P_{\max} > \begin{cases} \frac{1}{1-\sqrt[n]{\frac{6}{k\pi^2}}} & \text{if} \quad N \quad \text{is even} \\ \frac{1}{1-\sqrt[n]{\frac{8}{k\pi^2}}} & \text{if} \quad N \quad \text{is odd} \end{cases}$$

Proof. Considering that

$$\frac{n}{P_{\max}} < \sum_{i=1}^{n}\frac{1}{p_i} \text{ respective } \sum_{i=1}^{n}\frac{1}{p_i} < \frac{n}{P_{\min}}$$

from the theorem if follows the affirmation.

Remark 2. *Let* $N = p_1^{\alpha_1} p_2^{\alpha_2} ... p_n^{\alpha_n}$ *be a k-perfect number written in canonical form, then*

$$P_{\min} < \frac{2n}{k^2 - 1} + 2$$

(see the method of M. Perisastri's)

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