$\mathcal{K}$-manifolds locally described by Sasaki manifolds

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Abstract

$\mathcal{K}$-manifolds are normal metric globally framed f-manifolds whose Sasaki 2-form is closed. We introduce and study some subclasses of $\mathcal{K}$-manifolds. We describe some examples and we also state local decomposition theorems.

1 Introduction

Globally framed f-manifolds, also known as f-manifolds with parallelizable kernel ($f.pk$-manifolds), represent a natural generalization of almost contact manifolds ([8, 9]). Such manifolds have been studied by several authors and from different point of view ([1, 3, 4, 5, 7, 12]). They are manifolds M^{2n+s} equipped with an f-structure φ of rank $2n$ with kernel parallelizable by s vector fields $\xi_1, \ldots \xi_s$. Such manifolds always admit Riemannian metrics g which verify the compatibility condition $g(\varphi X, \varphi Y) = g(X,Y) - \sum_{i=1}^{s} \eta^i(X)\eta^i(Y)$, where $\eta^1, \ldots, \eta^s$ are the 1-forms dual to $\xi_1, \ldots \xi_s$. When the normality tensor field $N := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta^i \otimes \xi_i$, $[\varphi, \varphi]$ being the Nijenhuis torsion of φ, vanishes and the Sasaki 2-form $F = g(-, \varphi-)$ is closed one obtains a class of manifolds that generalizes quasi-Sasakian manifolds and are called $\mathcal{K}$-manifolds by D.E. Blair in [1]. Two special subclasses are also defined: $\mathcal{S}$-manifolds, by requiring that $d\eta^1 = \cdots = d\eta^s = F$, and $\mathcal{C}$-manifolds, by requiring that $d\eta^1 = \cdots = d\eta^s = 0$.

In this paper we study $\mathcal{K}$-manifolds M^{2n+s} subject either to the condition $\sum_{i=1}^{s} d\eta^i = F$ or to $\sum_{i=1}^{s} d\eta^i = 0$. Since for $s = 1$ the first case corresponds to the Sasaki manifolds and the second to cosymplectic manifolds, we shall consider $\mathcal{K}$-manifolds M^{2n+s} with $s \geq 2$. We denote by $\widehat{\mathcal{K}}$ the subclass of manifolds satisfying the first condition, which contains the products of Sasakian manifolds, and by $\mathcal{K}^0$ the subclass of those satisfying the second, which obviously contains $\mathcal{C}$-manifolds. We give an example of a $\mathcal{K}^0$-manifold non $\mathcal{C}$-manifold.

The most important results are local decomposition theorems: first we describe the local decomposition of an $\mathcal{S}$-manifold as Riemannian product of a $\sqrt{s}$-Sasakian manifold and a flat $(s-1)$-dimensional manifold. Then we prove that a $\widehat{\mathcal{K}}$-manifold M^{2n+s} with the property that there exists p, $1 \leq p < s$, such that $d\eta^i \neq 0$ for $i \leq p$ and $d\eta^i = 0$ for $i \geq p+1$, is locally a Riemannian product of a $\widehat{\mathcal{K}}$-manifold M_1^{2n+p} and an $(s-p)$-dimensional flat manifold. This allows to consider only $\widehat{\mathcal{K}}$-manifolds such that $d\eta^i \neq 0$ for each $i \in \{1, \dots, s\}$. We distinguish three subclasses: $\widehat{\mathcal{K}}_1$ if $d\eta^i = d\eta^j$ for any i, j; $\widehat{\mathcal{K}}_2$ if $d\eta^i \neq d\eta^j$ for any i, j; $\widehat{\mathcal{K}}_3$ if there exists $q \leq s-2$, $d\eta^i \neq d\eta^j$ for $i, j \leq q$ and $d\eta^i = d\eta^j$ for $i, j \geq q+1$. The $\widehat{\mathcal{K}}_1$-manifolds are strictly linked to $\mathcal{S}$-manifolds and we prove a local decomposition theorem (Theorem 4.1). After studying certain integrable and φ-invariant distributions, we are able to prove that $\widehat{\mathcal{K}}_2$-manifolds and $\widehat{\mathcal{K}}_3$-manifolds, verifying some hypotheses on the rank of the forms $\eta^1, \dots, \eta^s$, are locally product of s Sasakian manifolds (Theorem 5.2), in the first case, and of $q+1$ Sasakian manifolds and an $(s-q-1)$-dimensional flat manifold (Theorem 5.3) in the last case. Finally, we discuss the case $s = 2$ and the link with Vaisman manifolds.

All manifolds are assumed to be connected. We adopt the notation in [10] for the curvature tensor field.

2   Special types of $\mathcal{K}$-manifolds

It is well known, [3], that the Levi-Civita connection of a $\mathcal{K}$-manifold satisfies, for each $X, Y, Z \in \Gamma(TM)$,

$$g((\nabla_X \varphi)Y, Z) = \sum_{i=1}^{s} \left(d\eta^i(\varphi Y, X)\eta^i(Z) - d\eta^i(\varphi Z, X)\eta^i(Y) \right). \qquad (2.1)$$

In [6] we studied a class, here denoted by $\mathcal{K}^*$, of $\mathcal{K}$-manifolds with the property that $d\eta^i = 0$ for some $i \in \{1, \dots, s\}$ and $d\eta^j = F$ for the other values of the index, that is, up to the order, there exists p, $1 \leq p < s$ such that $d\eta^1 = \dots = d\eta^p = F$ and $d\eta^i = 0$ for $i \geq p+1$. We proved that such a $\mathcal{K}^*$-manifold can be viewed locally as a Riemannian product of an integral

submanifold M_1 of the distribution Im $\varphi \oplus < \xi_1, \ldots, \xi_p >$, which carries a structure of S-manifold of dimension $2n+p$, and an $(s-p)$-dimensional integral submanifold M_2 of the flat distribution $< \xi_{p+1}, \ldots, \xi_s >$. We begin with a local decomposition theorem for S-manifolds.

Theorem 2.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be an S-manifold. Then (M^{2n+s}, g) is locally a Riemannian product of a $\sqrt{s}$-Sasakian manifold and an $(s-1)$-dimensional flat manifold.*

Proof. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be an S-manifold. We put $\bar{\xi} = \sum_{i=1}^s \xi_i$ and $\bar{\eta} = \sum_{i=1}^s \eta^i$. Since $\bar{\xi} \in \ker \varphi$ we fix a basis $(\widetilde{\xi}_2, \ldots, \widetilde{\xi}_s)$ of $< \bar{\xi} >^\perp$ in $\ker \varphi$, $\widetilde{\xi}_j = \xi_1 - \xi_j$, $j \in \{2, \ldots, s\}$, and we obtain the orthogonal decomposition $TM = (\text{Im } \varphi \oplus < \bar{\xi} >) \oplus < \widetilde{\xi}_2, \ldots, \widetilde{\xi}_s >$. Moreover $\mathcal{D}_1 = \text{Im } \varphi \oplus < \bar{\xi} >$ and $\mathcal{D}_2 = < \widetilde{\xi}_2, \ldots, \widetilde{\xi}_s >$ are both integrable and totally geodesic distributions. Thus (M^{2n+s}, g) is locally a Riemannian product of integral submanifolds of $\mathcal{D}_1$ and $\mathcal{D}_2$, say M_1 and M_2. Clearly M_2 is $(s-1)$-dimensional and flat. On the other hand we normalize $\bar{\xi}$ obtaining $\widetilde{\xi} = \frac{\bar{\xi}}{\sqrt{s}}$, as $g(\bar{\xi}, \bar{\xi}) = s$. Then $(M_1, \varphi, \widetilde{\xi}, \widetilde{\eta}, g)$, $\widetilde{\eta} = \frac{1}{\sqrt{s}} \bar{\eta}$, is $(2n+1)$-dimensional, $\varphi(\widetilde{\xi}) = 0$, $g(\widetilde{\xi}, \widetilde{\xi}) = 1$ and $g(X, \widetilde{\xi}) = \widetilde{\eta}(X)$, so that $\widetilde{\eta}$ is the dual 1-form of $\widetilde{\xi}$. It is easy to verify that $\varphi^2 = -id + \widetilde{\eta} \otimes \widetilde{\xi}$ and $g(\varphi X, \varphi Y) = g(X, Y) - \widetilde{\eta}(X)\widetilde{\eta}(Y)$ so obtaining an almost contact metric structure $(\varphi, \widetilde{\xi}, \widetilde{\eta}, g)$ on M_1. The normality of the structure follows from $d\widetilde{\eta} \otimes \widetilde{\xi} = \frac{1}{\sqrt{s}} \sum_{i=1}^s d\eta^i \otimes \frac{1}{\sqrt{s}} \sum_{j=1}^s \xi_j = F \otimes \bar{\xi} = \sum_{i=1}^s d\eta^i \otimes \xi_i$ on M_1. Finally, for the Sasaki 2-form $\widetilde{F}$ of this structure on M_1, we obtain $d\widetilde{\eta} = \frac{1}{\sqrt{s}} d\bar{\eta} = \frac{1}{\sqrt{s}} \sum_{i=1}^s d\eta^i = \frac{1}{\sqrt{s}} sF = \sqrt{s}\widetilde{F}$. $\qquad \square$

Remark 2.1. We recall that an M^{2n+1} manifold admitting an α-Sasakian structure (φ, ξ, η, g) (i.e. with $d\eta = \alpha F$) carries also a Sasakian structure given by $(\varphi, \frac{\xi}{\alpha}, \alpha\eta, \alpha^2 g)$. Thus, by the above theorem, the S-manifold M^{2n+s} admits two foliations corresponding to the distributions $\mathcal{D}_1, \mathcal{D}_2$ and each leaf of the first admits the Sasaki structure $(\varphi, \frac{\bar{\xi}}{s}, \bar{\eta}, sg_{|M_1})$.

Corollary 2.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a $\mathcal{K}^*$-manifold with p, $1 \le p < s$, such that $d\eta^i = F$ for $i \le p$ and $d\eta^i = 0$ for $i \ge p + 1$. Then (M^{2n+s}, g) is locally the Riemannian product of a $\sqrt{p}$-Sasakian manifold and a flat $(s-1)$-dimensional manifold. Moreover, if $p \ge 2$, the manifold is also foliated by Sasaki manifolds and flat manifolds.*

Proof. Since (M^{2n+s}, g) is locally the Riemannian product of an S-manifold of dimension $2n + p$ and a flat $(s-p)$-dimensional manifold, we can apply Theorem 2.1 and Remark 2.1. $\qquad \square$

In the sequel we shall study metric $f.pk$-manifolds of dimension $2n + s$, $s \geq 2$, with a condition on the 1-form $\sum_{i=1}^s d\eta^i$. Precisely we consider the conditions

$$\text{a}) : \sum_{i=1}^s d\eta^i = F \qquad\qquad \text{b}) : \sum_{i=1}^s d\eta^i = 0. \qquad\qquad (2.2)$$

Actually, assuming the normality of the structure, $dF = 0$ (which is superfluous if a) holds) and $\eta^1 \wedge \ldots \wedge \eta^s \wedge F^n \neq 0$, we can consider two subclasses of $\mathcal{K}$-manifolds: the class $\widehat{\mathcal{K}}$ of manifolds satisfying a), and the class $\mathcal{K}^o$ of manifolds satisfying b).

The class $\widehat{\mathcal{K}}$ includes $\mathcal{K}^*$-manifolds with $p = 1$ and $s \geq 2$, while excludes $\mathcal{C}$-manifolds and $\mathcal{S}$-manifolds. One of the interest in studying this class comes from the fact that a finite product of Sasakian manifolds carries a structure of $\widehat{\mathcal{K}}$-manifold, as it will be stated in Theorem 2.2. On the other hand the class $\mathcal{K}^o$ includes $\mathcal{C}$-manifolds and excludes $\mathcal{S}$-manifolds.

From now on, $\sum_{i=1}^s \xi_i$ and $\sum_{i=1}^s \eta^i$ will be denoted by $\bar{\xi}$ and $\bar{\eta}$, respectively. Since in a $\mathcal{K}$-manifold $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, each ξ_i is Killing ([1]), we have immediately the following characterizations:

$$\begin{aligned} M^{2n+s} \ is \ a \ \widehat{\mathcal{K}} - manifold \ if \ and \ only \ if \quad \sum_{i=1}^s \nabla \xi_i = \nabla \bar{\xi} = -\varphi \\ M^{2n+s} \ is \ a \ \mathcal{K}^o - manifold \ if \ and \ only \ if \quad \sum_{i=1}^s \nabla \xi_i = \nabla \bar{\xi} = 0. \end{aligned} \quad (2.3)$$

From [5] we know that a $\mathcal{D}_a$-*homothetic deformation*, $a > 0$, of the $f.pk$-structure $(\varphi, \xi_i, \eta^i, g)$ on M^{2n+s} is a change of the structure tensors as follows:

$$\tilde{\varphi} = \varphi, \ \tilde{\eta}^i = a\eta^i, \ \tilde{\xi}_i = \frac{1}{a}\xi_i, \ 1 \leq i \leq s, \ \tilde{g} = ag + a(a-1)\sum_{j=1}^s \eta^j \otimes \eta^j$$

and one easily verifies the following result.

Proposition 2.1. *The classes $\widehat{\mathcal{K}}$ and $\mathcal{K}^o$ are both closed under $\mathcal{D}_a$-homothetic deformations.*

Now, we describe an example of $\mathcal{K}^o$-manifold which is not a $\mathcal{C}$-manifold.

Example 2.1. Let L be a real vector space of dimension $m \geq 4$ and $s \geq 2$ such that $m - s$ is an even number, say $2n$. In L we fix a basis

$$(Z_1, \ldots, Z_s, X_1, \ldots, X_n, Y_1, \ldots, Y_n) \qquad\qquad (2.4)$$

and define a Lie algebra structure putting the bracket $[-, -] = 0$ except for

$$[X_i, Y_i] = -[Y_i, X_i] = \sum_{t=1}^s a_i^t Z_t,$$

for any $i \in \{1, \ldots, n\}$, requiring that the matrix (a_i^t) of the type (s, n) has all its entries not zero and $\sum_{t=1}^{s} a_i^t = 0$, for any $i \in \{1, \ldots, n\}$.

Since $[L, L] = <Z_1, \ldots, Z_s>$ and $[L, [L, L]] = 0$, L is a nilpotent algebra. We consider on L the scalar product g_0 such that (2.4) is an orthonormal basis, the tensor φ_0 of the type $(1,1)$ and the 1-forms $\eta_0^1, \ldots \eta_0^s$ defined, for each $t, r \in \{1, \ldots, s\}$ and $i \in \{1, \ldots, n\}$, by:

$$\varphi_0(Z_t) = 0, \ \varphi_0(X_i) = Y_i, \ \varphi_0(Y_i) = -X_i, \ \eta_0^t(Z_r) = \delta_r^t, \ \eta_0^t(X_i) = \eta_0^t(Y_i) = 0.$$

Let G be the connected, simply connected Lie group having L as Lie algebra. It admits the structure $(\varphi, \xi_t, \eta^t, g)$ obtained by left-invariance from the structure $(\varphi_0, Z_t, \eta_0^t, g_0)$ on L. A direct computation yields that, for any $t \in \{1, \ldots, s\}$, $d\eta^t$ vanishes except on the left-invariant vector fields $(\tilde{X}_i, \tilde{Y}_i)$ determined by $X_i, Y_i, i \in \{1, \ldots, n\}$. In fact:

$$d\eta^t(\tilde{X}_i, \tilde{Y}_i) = -\frac{1}{2}\eta^t([\tilde{X}_i, \tilde{Y}_i]) = -\frac{1}{2}a_i^t \neq 0. \tag{2.5}$$

It is easy to check that $(G, \varphi, \xi_t, \eta^t, g)$ is a $\mathcal{K}$-manifold and (2.5) excludes the $\mathcal{C}$-structure. Finally, $\sum_{t=1}^{s} d\eta^t(\tilde{X}_i, \tilde{Y}_i) = -\frac{1}{2}\sum_{t=1}^{s} a_i^t = 0$, and G has a structure of $\mathcal{K}^0$-manifold.

The following result can be easily proved.

Theorem 2.2. *The product manifold $\widetilde{M} = \prod_{i=1}^{s} M_i$ of a family of almost contact metric manifolds $(M_i^{2n_i+1}, \varphi_i, \xi_i, \eta^i, g_i)$, $i \in \{1, \ldots, s\}$, with the structure defined by $\widetilde{\varphi} = \varphi_1 + \ldots + \varphi_s$, $\widetilde{\xi}_1 = \xi_1, \ldots, \widetilde{\xi}_s = \xi_s$, $\widetilde{\eta}^1 = \eta^1, \ldots, \widetilde{\eta}^s = \eta^s$ and $\widetilde{g} = g_1 + \ldots + g_s$, is a metric f.pk-manifold. Furthermore, the normality tensor field and the Sasaki 2-form of $\widetilde{M}$ are related to the analogous tensor fields of the factors by $\widetilde{N} = \sum_{i=1}^{s} N_i$ and $\widetilde{F} = \sum_{i=1}^{s} F_i$. Moreover*

1) *if each factor is a contact metric manifold then $\widetilde{M}$ satisfies (2.2)a),*

2) *if each factor is a Sasakian manifold then $\widetilde{M}$ is a $\widehat{\mathcal{K}}$-manifold,*

3) *if there exists $p \in \mathbb{N}$, $1 \leq p < s$ such that $M_1, \ldots, M_p$ are Sasakian and $M_{p+1}, \ldots, M_s$ are cosymplectic, then $\widetilde{M}$ is a $\widehat{\mathcal{K}}$-manifold.*

3   The $\widehat{\mathcal{K}}$-manifolds

Proposition 3.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a $\widehat{\mathcal{K}}$-manifold. Then we have*

1) $R_{\overline{\xi}X}Y = (\nabla_X\varphi)Y$, $R_{XY}\overline{\xi} = (\nabla_Y\varphi)X - (\nabla_X\varphi)Y$, $X, Y \in \Gamma(TM)$

2) $\nabla \bar{\eta} = F$

Proof. Since $\bar{\xi}$ is Killing, by (2.3), we get

$$(\nabla_X \varphi)Y = \nabla_X (\varphi Y) - \varphi \nabla_X Y = -\nabla_X (\nabla_Y \bar{\xi}) + \nabla_{\nabla_X Y} \bar{\xi} = -R_{X\bar{\xi}} Y = R_{\bar{\xi}X} Y.$$

The first Bianchi identity completes 1). A direct computation shows 2).   □

Next proposition implies that the class $\widehat{\mathcal{K}}$ is closed under finite products.

Proposition 3.2. *Let us suppose that* $(M_1^{2n+s}, \varphi_1, \xi_i, \eta^i, g_1)$, $i \in \{1, \ldots, s\}$, *and* $(M_2^{2m+t}, \varphi_2, \zeta_j, \omega^j, g_2)$, $j \in \{1, \ldots, t\}$, *are* $\widehat{\mathcal{K}}$-*manifolds. Then the Riemannian product* $(M_1^{2n+s} \times M_2^{2m+t}, \widetilde{g})$ *admits a* $\widehat{\mathcal{K}}$-*structure.*

Proof. On the product manifold $\widetilde{M}$, we consider the f-structure $\widetilde{\varphi} = \varphi_1 + \varphi_2$. Then it is easy to verify that $(\widetilde{\varphi}, \xi^i, \zeta^j, \eta^i, \omega^j, \widetilde{g})$, $i \in \{1, \ldots, s\}, j \in \{1, \ldots, t\}$ is the required structure, since $\widetilde{N} = N_1 + N_2$ and $\widetilde{F} = F_1 + F_2$.   □

Remark 3.1. In [4] it has been proved that no Einstein $\mathcal{S}$-manifold can exist. On the other hand we can construct an Einstein $\widehat{\mathcal{K}}$-manifold taking s Sasakian-Einstein manifolds M_i, with the same dimension $2n + 1$, and making their Riemannian product as in Theorem 2.2. It is well known that $Ric_i = \lambda g_i$, Ric_i being the Ricci tensor field on M_i, with Einstein constant $\lambda = 2n$, which is also the Einstein constant of the product.

Example 3.1. We consider the coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n, z^1, \ldots, z^n)$ on $\mathbb{R}^{3n}$, $n \geq 2$, and we put:

$$\xi_i = \frac{\partial}{\partial z^i}, \quad \eta^i = dz^i + 2y^i dx^i, \quad i \in \{1, \ldots, n\}$$

$$\varphi = \begin{pmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \quad g = \begin{pmatrix} Z & 0 & Y \\ 0 & I_n & 0 \\ Y & 0 & I_n \end{pmatrix}$$

where $Y = diag(2y^1, \ldots, 2y^n)$, $Z = diag(1 + 4(y^1)^2, \ldots, 1 + 4(y^n)^2)$. From long but easy computations we get that $\mathbb{R}^{3n}$ is a $\widehat{\mathcal{K}}$-manifold. In particular, we have $\eta^1 \wedge \ldots \wedge \eta^n \wedge F^n \neq 0$ and

$$d\eta^i = -2dx^i \wedge dy^i, \quad F = \sum_{i=1}^{n} (-2dx^i \wedge dy^i) = \sum_{i=1}^{n} d\eta^i.$$

Example 3.2. Another significant example can be obtained as a particular case of the example constructed in section 1.2 of [7].

Let G be a Lie group with Lie algebra $\mathfrak{g}$, $P \xrightarrow{\pi} M$ a G-principal fibre bundle and ω a connection on P. Moreover we suppose that there is given on G an almost contact metric structure $\mathcal{Z}_G$ determined by left-invariance of the following objects on $\mathfrak{g}$: a scalar product $g_{\mathfrak{g}}$, an almost contact structure $\varphi_{\mathfrak{g}}$, a vector ξ and a 1-form η satisfying the usual compatibility conditions. The induced Sasaki 2-form and the normality tensor field N_G on G are also left-invariant and they are determined by the corresponding bilinear maps $F_{\mathfrak{g}}$ and $N_{\mathfrak{g}}$ on $\mathfrak{g}$. We suppose that $(\varphi_M, \zeta, \mu, g_M)$ is an almost contact metric structure on M with Sasaki 2-form F_M and normality tensor field N_M.

The connection ω ensures for each $p \in P$ the splitting $T_p P = \mathcal{H}_p \oplus \mathcal{V}_p$. Via the well-known isomorphisms $\omega : \mathcal{V}_p \to \mathfrak{g}$, $d\pi : \mathcal{H}_p \to T_{\pi(p)} M$ we can define on TP an f-structure φ_P in such a way that the following diagram

$$
\begin{array}{ccccc}
T_p M & \xleftarrow{d\pi} & \mathcal{H}_p \oplus \mathcal{V}_p & \xrightarrow{\omega} & \mathfrak{g} \\
{\scriptstyle \varphi_M} \downarrow & & {\scriptstyle \varphi_P} \downarrow & & \downarrow {\scriptstyle \varphi_{\mathfrak{g}}} \\
T_p M & \xleftarrow{d\pi} & \mathcal{H}_p \oplus \mathcal{V}_p & \xrightarrow{\omega} & \mathfrak{g}
\end{array}
$$

commutes. Moreover, via the isomorphism $d\pi \times \omega : T_p P \to T_{\pi(x)} M \times \mathfrak{g}$ we define on P a Riemannian metric g_P, the vector fields $\widetilde{\xi}, \widetilde{\zeta}$, and the 1-forms $\widetilde{\eta}, \widetilde{\mu}$, obtaining on P a metric $f.pk$-structure $(\varphi_P, \widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}, \widetilde{\mu}, g_P)$ ([7]). Such a structure is G-invariant if and only if $(\varphi_{\mathfrak{g}}, \xi, \eta, g_{\mathfrak{g}})$ is Ad_G-invariant.

In [7] it is proved that if ω is flat and $(\varphi_M, \zeta, \mu, g_M)$, $(\mathfrak{g}, \varphi_{\mathfrak{g}}, \xi, \eta, g_{\mathfrak{g}})$ are normal, then the structure on P is normal.

Thus, assuming that ω is flat and $(M, \varphi_M, \zeta, \mu, g_M)$, $(\mathfrak{g}, \varphi_{\mathfrak{g}}, \xi, \eta, g_{\mathfrak{g}})$ are Sasaki, then $(P, \varphi_P, \widetilde{\xi}, \widetilde{\zeta}, \widetilde{\eta}, \widetilde{\mu}, g_P)$ is a $\widehat{\mathcal{K}}$-manifold. Namely, by Lemma 2.2 of [7] we get immediately the following identities

$$
d\widetilde{\eta} = d\eta \circ (\omega \wedge \omega), \quad d\widetilde{\mu} \;\; = \;\; \pi^*(d\mu), \quad F_P = F_{\mathfrak{g}} \circ (\omega \wedge \omega) + \pi^* F_M = d\widetilde{\eta} + d\widetilde{\mu},
$$

where F_P is the Sasaki 2-form associated to the $f.pk$-structure on P.

We state now a theorem which, in some way, reduces the study to the $\widehat{\mathcal{K}}$-manifolds such that $d\eta^i \neq 0$ for each $i \in \{1, \ldots, s\}$.

Theorem 3.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be a $\widehat{\mathcal{K}}$-manifold and suppose that there exists p, $1 \leq p < s$, such that $d\eta^i \neq 0$ for any $i \in \{1, \ldots, p\}$ and $d\eta^i = 0$ for any $i \geq p+1$. Then M^{2n+s} is locally a Riemannian product of a $\widehat{\mathcal{K}}$-manifold $(M_1^{2n+p}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, p\}$, and an $(s-p)$-dimensional flat manifold.*

Proof. We can consider the orthogonal decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$, where $\mathcal{D}_1 = \mathrm{Im}\,\varphi \oplus <\xi_1,\ldots,\xi_p>$ and $\mathcal{D}_2 = <\xi_{p+1},\ldots,\xi_s>$. Since the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are both integrable and totally geodesic, (M^{2n+s}, g) is locally a Riemannian product of their integral submanifolds, say M_1 and M_2. Clearly M_2 is $(s-p)$-dimensional and flat. It is easy to check that the induced structure $(\varphi, \xi_1, \ldots, \xi_p, \eta^1, \ldots, \eta^p, g)$ on M_1 is a $\mathcal{K}$-structure. Finally, for its Sasaki 2-form F' we have $F'(X,Y) = \sum_{i=1}^{s} d\eta^i(X,Y) = \sum_{i=1}^{p} d\eta^i(X,Y)$ and this completes the proof. $\qquad\square$

Owing to the above theorem, we shall study the $\widehat{\mathcal{K}}$-manifolds such that $d\eta^i \neq 0$ for each $i \in \{1,\ldots,s\}$. The following cases can occur:

1) $d\eta^i = d\eta^j$, for any i,j.

2) $d\eta^i \neq d\eta^j$, for any i,j.

3) There exists $q \leq s-2$ such that $d\eta^i \neq d\eta^j$, $i,j \leq q$ and $d\eta^i = d\eta^j$, $i,j \geq q+1$.

which give rise to three subclasses, denoted by $\widehat{\mathcal{K}}_1$, $\widehat{\mathcal{K}}_2$ and $\widehat{\mathcal{K}}_3$, respectively. In the last case, the first possibility is $d\eta^1 \neq d\eta^2 = d\eta^3$ and of course $s \geq 3$. Note that Example 3.1 and 3.2 belong to the class $\widehat{\mathcal{K}}_2$. Examples in the class $\widehat{\mathcal{K}}_1$ can be constructed starting from $\mathcal{S}$-manifolds, as described in next section.

4   The class $\widehat{\mathcal{K}}_1$

We denote by $\mathcal{G}(M^{2n+s}, \varphi, \xi_i, \eta^i)$ the set of the metric tensor fields compatible with the $f.pk$-structure (φ, ξ_i, η^i).

Proposition 4.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i)$, $i \in \{1,\ldots,s\}$, $s \geq 2$, be a normal $f.pk$-manifold such that $d\eta^1 = \ldots = d\eta^s \neq 0$. Then M^{2n+s} admits a metric $g \in \mathcal{G}(M^{2n+s}, \varphi, \xi_i, \eta^i)$ that makes M^{2n+s} an $\mathcal{S}$-manifold if and only if there exists $\widetilde{g} \in \mathcal{G}(M^{2n+s}, \varphi, \xi_i, \eta^i)$ that makes M^{2n+s} a $\widehat{\mathcal{K}}_1$-manifold.*

Proof. If M^{2n+s} is an $\mathcal{S}$-manifold, we can consider the new metric

$$\widetilde{g} = sg - (s-1)\sum_{i=1}^{s} \eta^i \otimes \eta^i\,. \tag{4.1}$$

Then, it is easy to check that $\widetilde{g} \in \mathcal{G}(M^{2n+s}, \varphi, \xi_i, \eta^i)$ and (2.2)a) holds. Vice versa let $\widetilde{g} \in \mathcal{G}(M^{2n+s}, \varphi, \xi_i, \eta^i)$ such that (2.2)a) holds. The transformation $g = \frac{1}{s}\widetilde{g} + \frac{(s-1)}{s}\sum_{i=1}^{s} \eta^i \otimes \eta^i$, gives $(M^{2n+s}, \varphi, \xi_i, \eta^i)$ a structure of $\mathcal{S}$-manifold. $\qquad\square$

Proposition 4.2. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i)$, $i \in \{1, \ldots, s\}$, $s \geq 2$, be an 𝒮-manifold and $\widetilde{g}$ the metric (4.1), that makes M^{2n+s} a $\widehat{\mathcal{K}}_1$-manifold. Then the Levi-Civita connections are tied by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{s-1}{s} \left(\overline{\eta}(Y)\varphi X + \overline{\eta}(X)\varphi Y \right), \qquad (4.2)$$

for each $X, Y \in \Gamma(TM)$. Moreover the curvature tensor fields are linked by

$$
\begin{aligned}
\widetilde{R}_{XY} Z = R_{XY} Z \quad &+ \quad \frac{s-1}{s} \Big(d\overline{\eta}(X,Z)\varphi Y - d\overline{\eta}(Y,Z)\varphi X + 2d\overline{\eta}(X,Y)\varphi Z \\
&+ \quad \overline{\eta}(Y)g(\varphi X, \varphi Z)\overline{\xi} - \overline{\eta}(X)g(\varphi Y, \varphi Z)\overline{\xi} \\
&+ \quad \frac{s+1}{s} \left(\overline{\eta}(Y)\overline{\eta}(Z)\varphi^2 X - \overline{\eta}(X)\overline{\eta}(Z)\varphi^2 Y \right) \Big), \qquad (4.3)
\end{aligned}
$$

for each $X, Y, Z \in \Gamma(TM)$.

Proof. The proofs are long but direct computations. We only notice that to obtain (4.2) we use $\eta^i(\widetilde{\nabla}_X Y) = X(\eta^i(Y)) - g(X, \varphi Y)$, $i \in \{1, \ldots, s\}$ and to prove (4.3) we use $\overline{\eta}(\widetilde{\nabla}_Y Z) = \overline{\eta}(\nabla_Y Z)$, $(\nabla_X \overline{\eta})Z = d\overline{\eta}(X,Z)$ and (4.2). $\square$

Theorem 4.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, \tilde{g})$ be a $\widehat{\mathcal{K}}_1$-manifold. Then $(M^{2n+s}, \tilde{g})$ is locally a Riemannian product of a $\frac{1}{\sqrt{s}}$-Sasakian manifold and an $(s-1)$-dimensional flat manifold.*

Proof. By Proposition 4.1 we know that there exists on M^{2n+s} an 𝒮-structure $(\varphi, \xi_i, \eta^i, g)$, and the metrics are linked by (4.1). As in Theorem 2.1 we have $TM = (\mathrm{Im}\,\varphi \oplus < \overline{\xi} >) \oplus < \widetilde{\xi}_2, \ldots \widetilde{\xi}_s >$, the distributions $\mathcal{D}_1 = \mathrm{Im}\,\varphi \oplus < \overline{\xi} >$ and $\mathcal{D}_2 = < \widetilde{\xi}_2, \ldots, \widetilde{\xi}_s >$ are both integrable, and one easily checks that they are parallel with respect to the Levi-Civita connection of $\tilde{g}$ and then totally geodesic. Thus $(M^{2n+s}, \widetilde{g})$ is locally a Riemannian product of the integral submanifolds M_1 and M_2 of $\mathcal{D}_1$ and $\mathcal{D}_2$. Clearly M_2 is $(s-1)$-dimensional and flat. Furthermore it is easy to verify that $(M_1, \varphi, \xi', \eta', g_1)$, $\eta' = \frac{1}{\sqrt{s}}\,\overline{\eta}$, $\xi' = \frac{1}{\sqrt{s}}\overline{\xi}$, $g_1 = \widetilde{g}_{|M_1}$ is a $(2n+1)$-dimensional normal contact metric manifold. We remark that $g_1 \neq sg_{|M_1}$ since $g_1(\xi', \xi') = 1$ while $sg(\xi', \xi') = s$. Finally, $F'(X,Y) = g_1(X, \varphi Y) = \widetilde{g}(X, \varphi Y) = sg(X, \varphi Y) = sF(X,Y) = d\overline{\eta} = \sqrt{s}d\eta'$ on M_1 and this completes the proof. $\square$

Remark 4.1. As in Remark 2.1 we have that the $\widehat{\mathcal{K}}_1$-manifold M^{2n+s} admits two foliations corresponding to the distributions $\mathcal{D}_1, \mathcal{D}_2$ and the leaves of the first admit a Sasaki structure $(\varphi, \overline{\xi}, \frac{\overline{\eta}}{s}, \frac{1}{s}g_1)$. This structure can be obtained from that given in the same remark, $(\varphi, \frac{\overline{\xi}}{s}, \overline{\eta}, sg)$, applying the $\mathcal{D}_a$-homothetic transformation with $a = \frac{1}{s}$.

Proposition 4.3. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a $\mathcal{K}$-manifold and assume that $d\eta^1 = \ldots = d\eta^s \neq 0$. Then M is a $\widehat{\mathcal{K}}_1$-manifold if and only if $\nabla \xi_i = -\frac{1}{s}\varphi$, for each $i \in \{1, \ldots, s\}$.*

Proof. Since each ξ_i is Killing then the condition $d\eta^i = d\eta^j$ is equivalent to $\nabla \xi_i = \nabla \xi_j$ and the result follows again from (2.3). $\qquad\square$

Proposition 4.4. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ be a $\widehat{\mathcal{K}}_1$-manifold. Then,*

$$(\nabla_X \varphi)Y = \frac{1}{s} \{g(\varphi Y, \varphi X)\overline{\xi} + \overline{\eta}(Y)\varphi^2 X\} \tag{4.4}$$

$$R_{XY}\overline{\xi} = \frac{1}{s} \{\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\} \tag{4.5}$$

$$K(X, \overline{\xi}) = \frac{1}{s}, \quad K(X, \xi_i) = \frac{1}{s^2}, \quad X \in \Gamma(\mathcal{D}). \tag{4.6}$$

Here K denotes the sectional curvature.

Proof. Since (2.2)a can be written as $F = s d\eta^1$, then (2.1) becomes

$$\begin{aligned}
g((\nabla_X \varphi)Y, Z) &= d\eta^1(\varphi Y, X)\overline{\eta}(Z) - d\eta^1(\varphi Z, X)\overline{\eta}(Y) \\
&= \frac{1}{s}\left(g(\varphi Y, \varphi X)g(Z, \overline{\xi}) + g(Z, \varphi^2 X)\overline{\eta}(Y)\right) \\
&= \frac{1}{s}\, g\left(g(\varphi Y, \varphi X)\overline{\xi} + \overline{\eta}(Y)\varphi^2 X, Z\right)
\end{aligned}$$

and we get (4.4). Now using Proposition 3.1 we obtain

$$R_{XY}\overline{\xi} = (\nabla_Y \varphi)X - (\nabla_X \varphi)Y = \frac{1}{s}\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2(X)\right).$$

Analogously, for any $i \in \{1, \ldots, s\}$ since $\nabla \xi_i = -\frac{1}{s}\varphi$ and ξ_i is Killing we have

$$R_{\xi_i X} Y = -\nabla_X(\nabla_Y \xi_i) + \nabla_{\nabla_X Y}\xi_i = \frac{1}{s}\left(\nabla_X \varphi Y - \varphi \nabla_X Y\right) = \frac{1}{s}\left(\nabla_X \varphi\right)Y$$

and then

$$R_{XY}\xi_i = \frac{1}{s}\left((\nabla_Y \varphi)X - (\nabla_X \varphi)Y\right) = \frac{1}{s}R_{XY}\overline{\xi} = \frac{1}{s^2}\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2(X)\right).$$

In particular, for any unit $X \in \Gamma(\mathcal{D})$ we have

$$R_{X\xi_i}\xi_i = \frac{1}{s^2}\left(-\varphi^2 X\right) = \frac{1}{s^2}\, X, \quad R_{X\overline{\xi}}\,\overline{\xi} = \frac{1}{s}\left(-\overline{\eta}(\overline{\xi})\varphi^2 X\right) = \frac{1}{s}\left(sX\right) = X$$

which give $K(X, \xi_i) = \frac{1}{s^2}$ and $K(X, \overline{\xi}) = \frac{1}{s}$. $\qquad\square$

5 Some results on the classes $\widehat{\mathcal{K}}_2$ and $\widehat{\mathcal{K}}_3$

We begin considering some distributions determined on a $\widehat{\mathcal{K}}$-manifold.

Lemma 5.1. *Let* $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, *be a* $\widehat{\mathcal{K}}$-*manifold. Hence for each* $i \in \{1, \ldots, s\}$, *the distributions* $\ker d\eta^i$ *and* $\ker \eta^i \cap \ker d\eta^i$ *are integrable and* φ-*invariant.*

Proof. Let be $X_1, X_2 \in \ker d\eta^i$ and $Y \in \Gamma(TM^{2n+s})$. Then, since $i_{X_1}(d\eta^i) = i_{X_2}(d\eta^i) = 0$, we have $0 = 3d^2\eta^i(X_1, X_2, Y) = -d\eta^i([X_1, X_2], Y)$, which implies $[X_1, X_2] \in \ker d\eta^i$. Now for any $X_1, X_2 \in \ker \eta^i \cap \ker d\eta^i$, we get $\eta^i([X_1, X_2]) = -2d\eta^i(X_1, X_2) = 0$ so that $[X_1, X_2] \in \ker \eta^i$. Furthermore, the φ-invariance follows easily, since the normality of the structure implies $d\eta^i(X, \xi_j) = 0$ and $d\eta^i(\varphi X, Y) = -d\eta^i(X, \varphi Y)$, for any $i, j \in \{1, \ldots, s\}$. $\quad\square$

Lemma 5.1 implies that, for each $i \in \{1, \ldots, s\}$, there exist two foliations determined by the distributions $\ker \eta^i \cap \ker d\eta^i$ and $\ker d\eta^i$. We notice that $<\xi_1, \ldots, \xi_s> = \ker \varphi \subseteq \ker d\eta^i$. Thus, if $d\eta^i = F$ for some $i \in \{1, \ldots, s\}$, then $\ker d\eta^i = \ker \varphi$ and $\ker \eta^i \cap \ker d\eta^i = <\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_s>$, which are both flat distributions. By the contrary, fixed $i \in \{1, \ldots, s\}$ with $d\eta^i \neq F$, which includes the possibility $d\eta^i = 0$, one easily obtains the following result.

Proposition 5.1. *Let* $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, *be a* $\widehat{\mathcal{K}}$-*manifold. Then, for any* $i \in \{1, \ldots, s\}$ *such that* $d\eta^i \neq F$, *the induced structure*

$$(\varphi, \xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_s, \eta^1, \ldots, \eta^{i-1}, \eta^{i+1}, \ldots, \eta^s, g)$$

on any integral submanifold N *of the distribution* $\ker \eta^i \cap \ker d\eta^i$ *turns out to be a* $\mathcal{K}$-*structure. Furthermore, any integral submanifold* $\widetilde{N}$ *of* $\ker d\eta^i$, *with the induced structure* $(\varphi, \xi_1, \ldots, \xi_s, \eta^1, \ldots, \eta^s, g)$, *becomes a* $\widehat{\mathcal{K}}$-*manifold. Obviously, if* $d\eta^i = 0$ *then* $\widetilde{N} = M^{2n+s}$.

Theorem 5.1. *Let* $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, *be a* $\widehat{\mathcal{K}}$-*manifold. For each* $j \in \{1, \ldots, s\}$, *the distribution*

$$D_j = \bigcap_{i \neq j} (\ker \eta^i \cap \ker d\eta^i), \tag{5.1}$$

is integrable and φ-*invariant. Moreover, for each* $j, h \in \{1, \ldots, s\}$, $j \neq h$, D_j *and* D_h *are orthogonal.*

Proof. Integrability and φ-invariance follow from Lemma 5.1. Let $X \in D_j$, $Y \in D_h$. Then for any $i \neq j$ we have $\eta^i(X) = 0$, $d\eta^i(X, -) = 0$, and, since $h \neq j$, $\eta^h(X) = 0$, $d\eta^h(X, -) = 0$. Analogously, for any $i \neq h$,

$\eta^i(Y) = 0$, $d\eta^i(Y, -) = 0$, in particular, $\eta^j(Y) = 0$, $d\eta^j(Y, -) = 0$. It follows that $\sum_{t=1}^{s} \eta^t(X)\eta^t(Y) = 0$ and $g(X, Y) = g(\varphi X, \varphi Y) = F(\varphi X, Y) = \sum_{t=1}^{s} d\eta^t(\varphi X, Y) = 0$. $\qquad\square$

Remark 5.1. If, for some $i \in \{1, \ldots, s\}$, $d\eta^i = F$, then $D_j = <\xi_j>$ for each $j \neq i$. Namely we have $\xi_j \in D_j$ and, being $D_j \subset \ker d\eta^i = <\xi_1, \ldots, \xi_s>$, for any $X \in D_j$ we get $X = \sum_{h=1}^{s} \alpha^h \xi_h$. Then, for any $h \neq j$, $\eta^h(X) = 0$ implies $\alpha^h = 0$ and $X \in <\xi_j>$.

Proposition 5.2. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, be a $\widehat{\mathcal{K}}_2$-manifold or a $\widehat{\mathcal{K}}_3$-manifold such that each η^t has rank $2k_t + 1$ and $k_1 + \cdots + k_s = n$. Then, for each $t \in \{1, \ldots, s\}$, $d\eta^t \neq F$.

Proof. In the given hypotheses, $F = d\eta^t$ for some index t implies $k_t = n$ and $d\eta^j = 0$, for any $j \neq t$, which is impossible. $\qquad\square$

5.1   A decomposition theorem for certain $\widehat{\mathcal{K}}_2$-manifolds

Proposition 5.3. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be a $\widehat{\mathcal{K}}_2$-manifold such that $rank(\eta^i) = 2k_i + 1$ and $k_1 + \cdots + k_s = n$. Then the integral submanifolds of any D_j inherit a structure of Sasakian manifold.

Proof. The hypotheses imply that for each $i \in \{1, \ldots, s\}$ $d\eta^i \neq F$. Let be $j \in \{1, \ldots, s\}$ and N an integral submanifold of D_j. Then surely $\xi_j \in \Gamma(TN)$ and $\xi_i \in \Gamma(TN^\perp)$ for any $i \neq j$. Furthermore, N has odd dimension since it is φ-invariant and the orthogonal complement of $<\xi_j>$ in TN verifies $<\xi_j>^\perp \subset \mathrm{Im}\varphi$. Then, by restriction and reduction, $\varphi' = (\varphi|_{<\xi_j>^\perp})_\#$ is an almost complex structure. Hence it is easy to check that the induced structure $(\varphi', \xi_j, \eta^j, g)$, is a Sasakian structure on N and $dimN = 2k_j + 1$. $\qquad\square$

Proposition 5.4. Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$ be a $\widehat{\mathcal{K}}_2$-manifold such that $rank(\eta^i) = 2k_i + 1$ and $k_1 + \cdots + k_s = n$. For each $j \in \{1, \ldots, s\}$, the distributions $\ker \eta^j \cap \ker d\eta^j$ and $D_j = \bigcap_{i \neq j}(\ker \eta^i \cap \ker d\eta^i)$ are orthogonal and complementary.

Proof. The two distributions are φ-invariant. They are also orthogonal since for $X \in D_j$ and $Y \in \ker \eta^j \cap \ker d\eta^j$ we have $F(\varphi X, Y) = \sum_{i=1}^{s} d\eta^i(\varphi X, Y) = 0$ and this implies $g(X, Y) = g(\varphi X, \varphi Y) = 0$. We choose a local φ-adapted basis of D_j, $\{\xi_j, e_1, \ldots, e_{k_j}, \varphi e_1, \ldots, \varphi e_{k_j}\}$ and we complete it to a φ-adapted basis of M^{2n+s} adding the vector fields

$$\{\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots \xi_s, e_{k_j+1}, \ldots, e_n, \varphi e_{k_j+1}, \ldots, \varphi e_n\}.$$

By the orthogonality it follows that such vector fields span the distribution $\ker \eta^j \cap \ker d\eta^j$. $\qquad\square$

Theorem 5.2. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$ be a $\widehat{\mathcal{K}}_2$-manifold such that $rank(\eta^i) = 2k_i + 1$ and $k_1 + \cdots + k_s = n$. Then M^{2n+s} is locally a product of s Sasakian manifolds.*

Proof. We argue by induction on s. Assume that $(M^{2n+2}, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ is a $\widehat{\mathcal{K}}_2$-manifold and that η^1, η^2 have rank $2k_1 + 1$ and $2k_2 + 1$ respectively, with $k_1 + k_2 = n$. Then $D_1 = \ker \eta^2 \cap \ker d\eta^2$ and $D_2 = \ker \eta^1 \cap \ker d\eta^1$ are integrable, φ-invariant, orthogonal and complementary. It follows that M^{2n+2} is locally a product of the integral submanifolds which, as in Proposition 5.3, inherit Sasakian structures. Now, assume that $s > 2$ and consider the integrable, φ-invariant, orthogonal and complementary distributions $D_s = \bigcap_{i \neq s}(\ker \eta^i \cap \ker d\eta^i)$ and $\ker \eta^s \cap \ker d\eta^s$. Thus M^{2n+s} is locally a product of integral submanifolds. The integral submanifolds of D_s are Sasakian manifolds and, as stated in Proposition 5.1, the integral submanifolds of $\ker \eta^s \cap \ker d\eta^s$ have a $\widehat{\mathcal{K}}$-structure which clearly satisfies the hypotheses. An application of the induction hypothesis completes the proof. $\square$

Remark 5.2. The above theorem applies to Example 3.1.

5.2   A decomposition theorem for certain $\widehat{\mathcal{K}}_3$-manifolds

Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be a $\widehat{\mathcal{K}}_3$-manifold. Let $d\eta^i \neq d\eta^j$ for each $i, j \in \{1, \ldots, q\}$ and $d\eta^i = d\eta^j$ for each $i, j \geq q + 1$, $q \leq s - 2$. Moreover, we assume that $rank(\eta^t) = 2k_t + 1$, $t \in \{1, \ldots, s\}$, and $k_1 + \cdots + k_s = n$, $q \leq s - 2$. Then it is easy to verify that Propositions 5.3 and 5.4 hold for any $j \leq q$ so we obtain the following result.

Proposition 5.5. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be a $\widehat{\mathcal{K}}_3$-manifold. Let $d\eta^i \neq d\eta^j$ for each $i, j \in \{1, \ldots, q\}$ and $d\eta^i = d\eta^j$ for each $i, j \geq q + 1$, $q \leq s - 2$. Moreover assume that $rank(\eta^t) = 2k_t + 1$, $t \in \{1, \ldots, s\}$, and $k_1 + \cdots + k_s = n$. Then one has*

  a) *For any $j \leq q$, the integral submanifolds of D_j inherit a structure of Sasakian manifold.*

  b) *For any $j \geq q + 1$, $D_j = < \xi_j >$ and its integral submanifolds are 1-dimensional and flat.*

Proof. Let be $j \in \{1, \ldots, q\}$. Arguing as in Proposition 5.3 we obtain a).
Now, fixed $j \geq q + 1$, we get that $\xi_j \in D_j$. Moreover for any $X \in D_j$ and any $Y \in \Gamma(TM^{2n+s})$, since $d\eta^j = d\eta^s$, we have $F(X, Y) = \sum_{t=1}^{s} d\eta^t(X, Y) = d\eta^j(X, Y) = d\eta^s(X, Y) = 0$ and this implies that $X \in \ker F = \ker \varphi$ that is $X = \sum_{t=1}^{s} \alpha^t \xi_t$. Then, being $\eta^t(X) = 0$ for any $t \neq j$, we obtain $X = \alpha^j \xi_j$ and $X \in < \xi_j >$, concluding the proof. $\square$

Theorem 5.3. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be a $\widehat{\mathcal{K}}_3$-manifold. Let $d\eta^i \neq d\eta^j$ for each $i, j \in \{1, \ldots, q\}$ and $d\eta^i = d\eta^j$ for each $i, j \geq q+1$, $q \leq s-2$. Moreover assume that $rank(\eta^t) = 2k_t + 1$, $t \in \{1, \ldots, s\}$, and $k_1 + \cdots + k_s = n$. Then M^{2n+s} is locally a product of $(q+1)$ Sasakian manifolds and a flat $(s-q-1)$-dimensional manifold.*

Proof. We argue by induction on q. If $q = 1$, the distributions D_1 and $\ker \eta^1 \cap \ker d\eta^1$ are integrable, φ-invariant, orthogonal and complementary, so, applying the above theorem and Proposition 5.1, M^{2n+s} is locally product of a Sasakian manifold and a $\widehat{\mathcal{K}}$-manifold which belongs to the class $\widehat{\mathcal{K}}_1$. Therefore by Remark 4.1, we obtain (locally) a product of two Sasakian manifolds and a flat $(s-2)$-dimensional manifold. Now we assume that $q \geq 2$. Thus considering the integrable, φ-invariant, orthogonal and complementary distributions $D_q = \bigcap_{i \neq q}(\ker \eta^i \cap \ker d\eta^i)$ and $\ker \eta^q \cap \ker d\eta^q$, M^{2n+s} turns out to be locally a product of integral submanifolds of such distributions. The integral submanifolds of D_q are Sasakian manifolds and, as stated in Proposition 5.1, the integral submanifolds of $\ker \eta^q \cap \ker d\eta^q$ have a $\widehat{\mathcal{K}}$-structure (actually $\widehat{\mathcal{K}}_3$) with $d\eta^i \neq d\eta^j$ for any $i, j \in \{1, \ldots, q-1\}$ and $d\eta^i = d\eta^j$ for any $i, j \in \{q+1, \ldots, s\}$. An application of the induction hypothesis completes the proof. $\qquad\square$

6 $\mathfrak{c}$-$\mathcal{K}$-manifolds, $\mathfrak{c} \in \mathbb{R}^s$.

Definition 6.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, \ldots, s\}$, be a $\mathcal{K}$-manifold and $\mathfrak{c} = (c^1, \ldots, c^s) \in \mathbb{R}^s$. $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is called a $\mathfrak{c}$-$\mathcal{K}$-manifold if for each $i \in \{1, \ldots, s\}$, $d\eta^i = c^i F$.*

Considering $\sum_{i=1}^s (c^i)^2$ one can distinguish two cases: $\sum_{i=1}^s (c^i)^2 = 0$, which corresponds to the $\mathcal{C}$-manifolds in [1], and $\sum_{i=1}^s (c^i)^2 \neq 0$.

Notice that

• If $c^i = 1$ for each $i \in \{1, \ldots, s\}$, we get the usual definition of $\mathcal{S}$-manifold.

• If there exists $p < s$ such that, up to the order, $c^i = 1$ for $i \in \{1, \ldots, p\}$ and $c^i = 0$ otherwise, we obtain the definition of $\mathcal{K}^*$-manifold.

• If $s = 1$, then $c = c^1 = 1$ and $c = c^1 \neq 1$ correspond to Sasaki and c-Sasaki manifolds, respectively.

• We also have $\sum_{i=1}^s d\eta^i = (\sum_{i=1}^s c^i)F$, so if $\sum_{i=1}^s c^i = 0$, the manifold belongs to the class $\mathcal{K}^0$, while if $\sum_{i=1}^s c^i = 1$ the manifold belongs to the class $\widehat{\mathcal{K}}$.

Now in the case: $\sum_{i=1}^s c^i \neq 0$ we prove a local decomposition theorem.

Theorem 6.1. *Let $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$, be a $\mathfrak{c}$-$\mathcal{K}$-manifold, $\sum_{i=1}^s c^i \neq 0$. Then (M^{2n+s}, g) is locally a Riemannian product of a $\sqrt{\alpha}$-Sasakian manifold, $\alpha = \sum_{i=1}^s (c^i)^2$, and an $(s-1)$-dimensional flat manifold.*

Proof. We put $\overline{\xi} = \sum_{i=1}^{s} c^i \xi_i$ and $\overline{\eta} = \sum_{i=1}^{s} c^i \eta^i$. Since $\overline{\xi} \in \ker \varphi$, we can consider a basis $(\overline{\xi}_2, \ldots, \overline{\xi}_s)$ of $< \overline{\xi} >^{\perp}$ in $\ker \varphi$, so that we obtain the orthogonal decomposition $TM = (\text{Im } \varphi \oplus < \overline{\xi} >) \oplus < \overline{\xi}_2, \ldots, \overline{\xi}_s >$. Namely, for a $\zeta \in \ker \varphi$ such that $g(\zeta, \overline{\xi}) = 0$, we have $\zeta = \sum_{j=1}^{s} \lambda^j \xi_j$ and $0 = g(\sum_{j=1}^{s} \lambda^j \xi_j, \sum_{i=1}^{s} c^i \xi_i) = \sum_{i=1}^{s} \lambda^i c^i$. Thus for any $h \in \{2, \ldots, s\}$ there exist constants β_h^i such that $\overline{\xi}_h = \sum_{i=1}^{s} \beta_h^i \xi_i$. Moreover, the dual 1-forms are given by $\overline{\eta}^h = \sum_{i=1}^{s} \beta_h^i \eta^i$ and it is easy to check that $d\overline{\eta}^h = 0$, from which $\nabla \overline{\xi}_h = 0$ follows.

Since $\mathcal{D}_1 = \text{Im } \varphi \oplus < \overline{\xi} >$ and $\mathcal{D}_2 = < \overline{\xi}_2, \ldots, \overline{\xi}_s >$ are both integrable, totally geodesic distributions, then (M^{2n+s}, g) is locally a Riemannian product of integral submanifolds M_1 of $\mathcal{D}_1$ and M_2 of $\mathcal{D}_2$. Clearly M_2 is $(s-1)$-dimensional and flat. Now, we normalize $\overline{\xi}$ obtaining $\widetilde{\xi} = \frac{\overline{\xi}}{\sqrt{\alpha}}$, $\alpha = \sum_{i=1}^{s} (c^i)^2$. Then $(M_1, \varphi, \widetilde{\xi}, \widetilde{\eta}, g)$, $\widetilde{\eta} = \frac{1}{\sqrt{\alpha}} \overline{\eta}$, is $(2n+1)$-dimensional. One easily checks that $(M_1, \varphi, \widetilde{\xi}, \widetilde{\eta}, g)$ is an almost contact metric manifold and the normality follows from $d\widetilde{\eta} \otimes \widetilde{\xi} = \frac{1}{\sqrt{\alpha}} \sum_{i=1}^{s} c^i d\eta^i \otimes \frac{1}{\sqrt{\alpha}} \sum_{j=1}^{s} c^j \xi_j = F \otimes \overline{\xi} = \sum_{i=1}^{s} d\eta^i \otimes \xi_i$ on M_1. Finally, $d\widetilde{\eta} = \frac{1}{\sqrt{\alpha}} d\overline{\eta} = \frac{1}{\sqrt{\alpha}} \sum_{i=1}^{s} c^i d\eta^i = \frac{1}{\sqrt{\alpha}} \sum_{i=1}^{s} (c^i)^2 F = \sqrt{\alpha} \widetilde{F}$. This completes the proof. $\qquad \square$

As corollaries we obtain Theorem 2.1 and its corollary.

Remark 6.1. Suppose that $(M^{2n+s}, \varphi, \xi_i, \eta^i, g)$ is a $\mathfrak{c}$-$\mathcal{K}$-manifold such that $\sum_{i=1}^{s} c^i = 1$. Then $\sum_{i=1}^{s} d\eta^i = F$ and the manifold turns out to be a $\widehat{\mathcal{K}}$-manifold. After an application of Theorem 3.1, if necessary i.e. if some $c^i = 0$, we have that M^{2n+s} or its non flat factor $(M_1^{2n+p}, \varphi, \xi_i, \eta^i, g)$, $(\sum_{i=1}^{p} c^i = 1)$, falls in one of the classes $\widehat{\mathcal{K}}_1$, $\widehat{\mathcal{K}}_2$, $\widehat{\mathcal{K}}_3$. Let us suppose that they belong to $\widehat{\mathcal{K}}_1$. Then, by Theorem 4.1, (M^{2n+s}, g) is locally a Riemannian product of a $\frac{1}{\sqrt{s}}$-Sasakian manifold and an $(s-1)$-dimensional flat manifold. On the other hand, from $\sum_{i=1}^{s} d\eta^i = F$ and $d\eta^1 = \ldots = d\eta^s$ it follows $d\eta^i = \frac{1}{s} F$, $c^i = \frac{1}{s}$ for each $i \in \{1, \ldots, s\}$ and $\alpha = \sum_{i=1}^{s} (c^i)^2 = \frac{1}{s}$, according to Theorem 6.1. Finally we remark that being each η^i of rank $2n+1$, Theorems 5.2 and 5.3 do not apply to these manifolds.

7 $\widehat{\mathcal{K}}$-manifolds of dimension $2n+2$

We begin proving the following result.

Proposition 7.1. *Let* $(M^{2n+s}, \varphi, \xi_i, \eta^i)$, $i \in \{1, \ldots, s\}$, *be a normal f.pk-manifold such that* $d\eta^1 = \ldots = d\eta^p \neq 0$, $1 \leq p < s$, *and* $d\eta^i = 0$ *for any* $i \geq p+1$. *Then* M^{2n+s} *admits a metric* $g \in \mathcal{G}(M^{2n+s}, \varphi, \xi_i, \eta^i)$ *that makes*

M^{2n+s} a $\mathcal{K}^*$-manifold with $F = d\eta^1 = \ldots = d\eta^p$ if and only if there exists a metric $\widetilde{g} \in \mathcal{G}(M^{2n+s}, \varphi, \xi_i, \eta^i)$ that makes M^{2n+s} a $\widehat{\mathcal{K}}$-manifold.

Proof. The proof goes on as in Proposition 4.1, simply considering the metric $\widetilde{g} = pg - (p-1)\sum_{i=1}^{s} \eta^i \otimes \eta^i$. Finally, $\widetilde{F} = pF = \sum_{i=1}^{p} d\eta^i = \sum_{i=1}^{s} d\eta^i$. $\square$

Theorem 7.1. *Let $(M^{2n+2}, \varphi, \xi_i, \eta^i)$, $i \in \{1, 2\}$, be a normal f.pk-manifold and suppose that $d\eta^1 \neq 0$ and $d\eta^2 = 0$. Then the following assertions are equivalent:*

(a) *there exists a metric $g \in \mathcal{G}(M^{2n+2}, \varphi, \xi_i, \eta^i)$ that makes M^{2n+2} a $\mathcal{K}$-manifold with $F = d\eta^1$,*

(b) *(M^{2n+2}, g) carries a structure of Vaisman manifold.*

Proof. From Proposition 7.1 it follows that the metric g makes M^{2n+2} a $\widehat{\mathcal{K}}$-manifold. Then, the equivalence between (a) and (b) is proved in [11, 6] and the links between the two structures are given by

$$\xi_2 = B, \ \xi_1 = JB, \ \eta^2 = \omega, \ \eta^1 = -\omega \circ J, \ \varphi = J + \eta^1 \otimes \xi_2 - \eta^2 \otimes \xi_1,$$

where B is the unit Lee vector field and ω the Lee form. $\square$

Now, fixed a $\widehat{\mathcal{K}}$-manifold $(M^{2n+2}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, 2\}$, with $d\eta^1 \neq 0$ and $d\eta^2 \neq 0$, we have the following possibilities: $d\eta^1 \neq d\eta^2 \quad or \quad d\eta^1 = d\eta^2$, which means that the manifold belongs to the class $\widehat{\mathcal{K}}_2$, $\widehat{\mathcal{K}}_1$, respectively.

In the first case, $d\eta^1$ and $d\eta^2$ are both different from F, otherwise, being $d\eta^1 + d\eta^2 = F$, one of them must vanish. Hence Theorem 5.2 ensures that the manifold is locally product of two Sasakian manifolds.

In the last case, by Theorem 4.1 we know that the manifold is locally a Riemannian product of a $\frac{1}{\sqrt{2}}$-Sasakian manifold and a 1-dimensional flat manifold. Moreover, we have:

Proposition 7.2. *Let $(M^{2n+2}, \varphi, \xi_i, \eta^i, g)$, $i \in \{1, 2\}$ be a $\widehat{\mathcal{K}}$-manifold such that $d\eta^1 = d\eta^2$, then M^{2n+2} admits a Vaisman structure.*

Proof. Let us put $\widetilde{\eta}^1 = \frac{\eta^1 + \eta^2}{\sqrt{2}}$, $\widetilde{\eta}^2 = \frac{\eta^1 - \eta^2}{\sqrt{2}}$, $\widetilde{\xi}_1 = \frac{\xi_1 + \xi_2}{\sqrt{2}}$, $\widetilde{\xi}_2 = \frac{\xi_1 - \xi_2}{\sqrt{2}}$. It is easy to verify that $(M^{2n+2}, \varphi, \widetilde{\xi}_i, \widetilde{\eta}^i, g)$, $i \in \{1, 2\}$ is a normal metric f.pk-manifold and that $d\widetilde{\eta}^1 = \frac{1}{\sqrt{2}} F$, $d\widetilde{\eta}^2 = 0$. Here F is the Sasaki 2-form of both the structures on M^{2n+2}. It is well known ([9]) that $J = \varphi - \widetilde{\eta}^1 \otimes \widetilde{\xi}_2 + \widetilde{\eta}^2 \otimes \widetilde{\xi}_1$ gives (M^{2n+2}, g) a Hermitian structure with Kähler form $\Omega = F + \widetilde{\eta}^1 \wedge \widetilde{\eta}^2$. Hence $d\Omega = \frac{1}{\sqrt{2}} F \wedge \widetilde{\eta}^2 = \frac{1}{\sqrt{2}} \widetilde{\eta}^2 \wedge \Omega$, that is $\omega = \frac{1}{\sqrt{2}} \widetilde{\eta}^2$ is the Lee form. $\square$

An interesting example related to the above proposition arises from the S-structure on the 4-dimensional manifold $U(2)$ described in [7]. On $U(2)$, one considers the left-invariant vector fields, ξ_1, ξ_2, X, Y, determined, in the same order, by the following basis of the Lie algebra $\mathfrak{u}(2)$:

$$\imath E_{11}, \ -\imath E_{22}, \ E_{12} - E_{21}, \ \imath(E_{12} + E_{21})$$

where $(E_{ij})_{i,j \in \{1,2\}}$ is the canonical basis of $gl(2, \mathbb{C})$. Then,

$$[X, Y] = 2\xi_1 + 2\xi_2, \quad [X, \xi_i] = -Y, \quad [Y, \xi_i] = X, \quad [\xi_i, \xi_j] = 0$$

for any $i, j \in \{1, 2\}$. One considers the left-invariant 1-forms η^1 and η^2 determined by the dual 1-forms of $\imath E_{11}$ and $-\imath E_{22}$, respectively, a left-invariant tensor field φ such that $\varphi(X) = Y$, $\varphi(Y) = -X$ and $\varphi(\xi_1) = \varphi(\xi_2) = 0$ and a left-invariant metric g such that the vector fields ξ_1, ξ_2, X, Y form an orthonormal basis. Then $(U(2), \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ becomes an S-manifold. Hence, by Proposition 4.1 we obtain a $\widehat{\mathcal{K}}_1$-structure with the new metric $\widetilde{g} = 2g - \eta^1 \otimes \eta^1 - \eta^2 \otimes \eta^2$ and we can apply Proposition 7.2.

Theorem 7.2. *Let M be a $(2n+2)$-dimensional manifold. Then the following propositions are equivalent*

(a) *M admits a $\widehat{\mathcal{K}}$-structure $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ with $0 \neq d\eta^1 \neq d\eta^2 \neq 0$*

(b) *M admits a $\mathcal{K}$-structure $(\varphi, \widetilde{\xi}_1, \widetilde{\xi}_2, \widetilde{\eta}^1, \widetilde{\eta}^2, g)$ such that the Sasaki 2-form F verifies $F = \alpha \, d\widetilde{\eta}^1 + \beta \, d\widetilde{\eta}^2$, where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 2$.*

Proof. Let us assume *(a)*. Then we take $\theta \in [0, 2\pi[$ and put

$$\begin{aligned} \widetilde{\xi}_1 &= \cos\theta \, \xi_1 + \sin\theta \, \xi_2, \quad \widetilde{\xi}_2 = -\sin\theta \, \xi_1 + \cos\theta \, \xi_2 \\ \widetilde{\eta}^1 &= \cos\theta \, \eta^1 + \sin\theta \, \eta^2, \quad \widetilde{\eta}^2 = -\sin\theta \, \eta^1 + \cos\theta \, \eta^2. \end{aligned}$$

Clearly, one has $g(X, \widetilde{\xi}_1) = \widetilde{\eta}^1(X)$, $g(X, \widetilde{\xi}_2) = \widetilde{\eta}^2(X)$, for any $X \in \Gamma(TM)$. Furthermore, from $\widetilde{\eta}^1(X)\widetilde{\xi}_1 + \widetilde{\eta}^2(X)\widetilde{\xi}_2 = \eta^1(X)\xi_1 + \eta^2(X)\xi_2$ it follows that $\varphi^2(X) = -X + \widetilde{\eta}^1(X)\widetilde{\xi}_1 + \widetilde{\eta}^2(X)\widetilde{\xi}_2$. Analogously the compatibility of the metric can be proved observing that $\widetilde{\eta}^1 \otimes \widetilde{\eta}^1 + \widetilde{\eta}^2 \otimes \widetilde{\eta}^2 = \eta^1 \otimes \eta^1 + \eta^2 \otimes \eta^2$ and the normality of the structure follows by $d\widetilde{\eta}^1 \otimes \widetilde{\xi}_1 + d\widetilde{\eta}^2 \otimes \widetilde{\xi}_2 = d\eta^1 \otimes \xi_1 + d\eta^2 \otimes \xi_2$. One obtains $F = d\eta^1 + d\eta^2 = (\cos\theta + \sin\theta)d\widetilde{\eta}^1 + (\cos\theta - \sin\theta)d\widetilde{\eta}^2$, since $\eta^1 = \cos\theta \, \widetilde{\eta}^1 - \sin\theta \, \widetilde{\eta}^2$, $\eta^2 = \sin\theta \, \widetilde{\eta}^1 + \cos\theta \, \widetilde{\eta}^2$. Hence, $(M, \varphi, \widetilde{\xi}_1, \widetilde{\xi}_2, \widetilde{\eta}^1, \widetilde{\eta}^2, g)$ is a $\mathcal{K}$-manifold and *(b)* follows by putting $\alpha = \cos\theta + \sin\theta$ and $\beta = \cos\theta - \sin\theta$.

Now assuming *(b)*, $\alpha^2 + \beta^2 = 2$ implies $\frac{(\alpha+\beta)^2}{4} + \frac{(\alpha-\beta)^2}{4} = 1$, so that one can put $\cos\theta = \frac{\alpha+\beta}{2}$, $\sin\theta = \frac{\alpha-\beta}{2}$. It follows that $\alpha = \cos\theta + \sin\theta$, $\beta = \cos\theta - \sin\theta$

and $F = \alpha\, d\widetilde{\eta}^1 + \beta\, d\widetilde{\eta}^2 = (\cos\theta + \sin\theta)d\widetilde{\eta}^1 + (\cos\theta - \sin\theta)d\widetilde{\eta}^2$. Then, putting

$$
\begin{aligned}
\eta^1 &= \cos\theta\ \widetilde{\eta}^1 - \sin\theta\ \widetilde{\eta}^2 & \eta^2 &= \sin\theta\ \widetilde{\eta}^1 + \cos\theta\ \widetilde{\eta}^2 \\
\xi_1 &= \cos\theta\ \widetilde{\xi}_1 - \sin\theta\ \widetilde{\xi}_2 & \xi_2 &= \sin\theta\ \widetilde{\xi}_1 + \cos\theta\ \widetilde{\xi}_2
\end{aligned}
$$

one verifies that $(\varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$ is a $\mathcal{K}$-structure and $F = d\eta^1 + d\eta^2$. $\quad\square$

Remark 7.1. The family of $\mathcal{K}$-structures described in condition (b) is parameterized on the sphere S^1 of radius $\sqrt{2}$. In particular for $\theta \in \{0, \frac{1}{2}\pi, \pi, \frac{3}{2}\pi\}$ one obtains the structure given in (a) and those obtained reversing (ξ_1, η^1) and/or (ξ_2, η^2).

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