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# Monomial ideals of minimal depth

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**Abstract**

Let $S$ be a polynomial algebra over a field. We study classes of monomial ideals (as for example lexsegment ideals) of $S$ having minimal depth. In particular, Stanley's conjecture holds for these ideals. Also we show that if $I$ is a monomial ideal with $\mathrm{Ass}(S/I) = \{P_1, P_2, \dots, P_s\}$ and $P_i \not\subset \sum_{1=j\neq i}^s P_j$ for all $i \in [s]$, then Stanley's conjecture holds for $S/I$.

## Introduction

Let $K$ be a field and $S = K[x_1, \dots, x_n]$ be a polynomial ring in $n$ variables over $K$. Let $I \subset S$ be a monomial ideal and $I = \cap_{i=1}^s Q_i$ an irredundant primary decomposition of $I$, where the $Q_i$ are monomial ideals. Let $Q_i$ be $P_i$-primary. Then each $P_i$ is a monomial prime ideal and $\mathrm{Ass}(S/I) = \{P_1, \dots, P_s\}$.

According to Lyubeznik [9] the size of $I$, denoted $\mathrm{size}(I)$, is the number $a + (n - b) - 1$, where $a$ is the minimum number $t$ such that there exist $j_1 < \cdots < j_t$ with

$$\sqrt{\sum_{l=1}^{t} Q_{j_l}} = \sqrt{\sum_{j=1}^{s} Q_j},$$

and where $b = \mathrm{ht}(\sum_{j=1}^s Q_j)$. It is clear from the definition that $\mathrm{size}(I)$ depends only on the associated prime ideals of $S/I$. In the above definition if we replace "there exists $j_1 < \cdots < j_t$" by "for all $j_1 < \cdots < j_t$", we obtain the definition of $\mathrm{bigsize}(I)$, introduced by Popescu [11]. Clearly $\mathrm{bigsize}(I) \geq \mathrm{size}(I)$.

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**Theorem 0.1.** *(Lyubeznik [9]) Let $I \subset S$ ba a monomial ideal then* $\operatorname{depth}(I) \geq 1 + \operatorname{size}(I)$.

Herzog, Popescu and Vladoiu say in [5] that a monomial ideal $I$ has *minimal depth*, if $\operatorname{depth}(I) = \operatorname{size}(I) + 1$. Suppose above that $P_i \not\subset \sum_{1=j\neq i}^{s} P_j$ for all $i \in [s]$. Then $I$ has minimal depth as shows our Corollary 1.3 which extends [11, Theorem 2.3]. It is easy to see that if $I$ has bigsize 1 then it must have minimal depth (see our Corollary 1.5).

Next we consider the lexicographical order on the monomials of $S$ induced by $x_1 > x_2 > \cdots > x_n$. Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree $d$ of $S$. For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{lex} v$, the set

$$\mathcal{L}(u,v) = \{w \in \mathcal{M}_d | u \geq_{lex} w \geq_{lex} v\}$$

is called a lexsegment set. A lexsegment ideal in $S$ is a monomial ideal of $S$ which is generated by a lexsegment set. We show that a lexsegment ideal has minimal depth (see our Theorem 1.6).

Now, let $M$ be a finitely generated multigraded $S$-module, $z \in M$ be a homogeneous element in $M$ and $zK[Z]$, $Z \subseteq \{x_1, \ldots, x_n\}$ the linear $K$-subspace of $M$ of all elements $zf$, $f \in K[Z]$. Such a linear $K$-subspace $zK[Z]$ is called a Stanley space of dimension $|Z|$ if it is a free $K[Z]$-module, where $|Z|$ denotes the number of indeterminates in $Z$. A presentation of $M$ as a finite direct sum of spaces $\mathcal{D}: \ M = \bigoplus_{i=1}^{r} z_i K[Z_i]$ is called a Stanley decomposition. Stanley depth of a decomposition $\mathcal{D}$ is the number

$$\operatorname{sdepth} \mathcal{D} = \min\{|Z_i| : i = 1, \ldots, r\}.$$

The number

$$\operatorname{sdepth}(M) := \max\{\operatorname{sdepth}(\mathcal{D}) : \text{Stanley decomposition of } M\}$$

is called Stanley depth of $M$. In [14] R. P. Stanley conjectured that

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M).$$

**Theorem 0.2** ([5])**.** *Let $I \subset S$ be a monomial ideal then* $\operatorname{sdepth}(I) \geq 1 + \operatorname{size}(I)$. *In particular, Stanley's conjecture holds for the monomial ideals of minimal depth.*

As a consequence, Stanley's depth conjecture holds for all ideals considered above since they have minimal depth. It is still not known a relation between $\operatorname{sdepth}(I)$ and $\operatorname{sdepth}(S/I)$, but our Theorem 2.3 shows that Stanley's conjecture holds also for $S/I$ if $P_i \not\subset \sum_{1=j\neq i}^{s} P_j$ for all $i \in [s]$. Some of the recent development about the Stanley's conjecture is given in [12].

## 1 Minimal depth

We start this section extending some results of Popescu in [11]. Lemma 1.1, Proposition 1.2, Lemma 1.4 and Corollary 1.5 were proved by Popescu when $I$ is a squarefree monomial ideal. We show that with some small changes the same proofs work even in the non-squarefree case.

**Lemma 1.1.** *Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of $I$ as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_s \not\subset \sum_{i=1}^{s-1} P_i$, then*

$$\operatorname{depth}(S/I) = \min\{\operatorname{depth}(S/\cap_{i=1}^{s-1}Q_i), \operatorname{depth}(S/Q_s),\ 1+\operatorname{depth}(S/\cap_{i=1}^{s-1}(Q_i+Q_s))\}.$$

*Proof.* We have the following exact sequence

$$0 \longrightarrow S/I \longrightarrow S/\cap_{i=1}^{s-1} Q_i \oplus S/Q_s \longrightarrow S/\cap_{i=1}^{s-1}(Q_i+Q_s) \longrightarrow 0.$$

Clearly $\operatorname{depth}(S/I) \le \operatorname{depth}(S/Q_s)$ by [1, Proposition 1.2.13]. Choosing $x_j^a$ where $x_j \in P_s \not\subset \sum_{i=1}^{s-1} P_i$ and $a$ is minimum such that $x_j^a \in Q_s$ we see that $I : x_j^a = \cap_{i=1}^{s-1} Q_i$ and by [13, Corollary 1.3] we have

$$\operatorname{depth}(S/I) \le \operatorname{depth} S/(I : x_j^a) = \operatorname{depth} S/(\cap_{i=1}^{s-1}Q_i).$$

Now by using Depth Lemma (see [15, Lemma 1.3.9]) we have

$$\operatorname{depth}(S/I) = \min\{\operatorname{depth}(S/\cap_{i=1}^{s-1}Q_i), \operatorname{depth}(S/Q_s),\ 1+\operatorname{depth}(S/\cap_{i=1}^{s-1}(Q_i+Q_s))\},$$

which is enough. □

**Proposition 1.2.** *Let $I = \bigcap_{i=1}^{s} Q_i$ be the irredundant presentation of $I$ as an intersection of primary monomial ideals. Let $P_i := \sqrt{Q_i}$. If $P_i \not\subset \sum_{1=i\neq j}^{s-1} P_j$ for all $i \in [s]$. Then* $\operatorname{depth}(S/I) = s-1$.

*Proof.* It is enough to consider the case when $\sum_{j=1}^{s} P_j = \mathfrak{m}$. We use induction on $s$. If $s = 1$ the result is trivial. Suppose that $s > 1$. By Lemma 1.1 we get

$$\operatorname{depth}(S/I) = \min\{\operatorname{depth}(S/\cap_{i=1}^{s-1}Q_i), \operatorname{depth}(S/Q_s),\ 1+\operatorname{depth}(S/\cap_{i=1}^{s-1}(Q_i+Q_s))\}.$$

Then by induction hypothesis we have

$$\operatorname{depth}(S/\cap_{i=1}^{s-1} Q_i) = s-2+\dim(S/(\sum_{i=1}^{s-1} Q_i)) \ge s-1.$$

We see that $\cap_{i=1}^{s-1}(Q_i + Q_s)$ satisfies also our assumption, the induction hypothesis gives $\operatorname{depth}(S/\cap_{i=1}^{s-1}(Q_i+Q_s)) = s-2$. Since $Q_i \not\subset Q_s$, $i < s$ by our assumption we get $\operatorname{depth}(S/Q_s) > \operatorname{depth}(S/(Q_i+Q_s))$ for all $i < s$. It follows $\operatorname{depth}(S/Q_s) \geq 1 + \operatorname{depth}(S/\cap_{i=1}^{s-1}(Q_i+Q_s))$ which is enough. $\square$

**Corollary 1.3.** *Let $I \subset S$ be a monomial ideal such that $\operatorname{Ass}(S/I) = \{P_1 \ldots, P_s\}$ where $P_i \not\subset \sum_{1=j\neq i}^{s} P_j$ for all $i \in [s]$. Then $I$ has minimal depth.*

*Proof.* Clearly $\operatorname{size}(I) = s-1$ and by Proposition 1.2 we have $\operatorname{depth}(I) = s$, thus we have $\operatorname{depth}(I) = \operatorname{size}(I)+1$, i.e. $I$ has minimal depth. $\square$

**Lemma 1.4.** *Let $I = \cap_{i=1}^{s} Q_i$ be the irredundant primary decomposition of $I$ and $\sqrt{Q_i} \neq \mathfrak{m}$ for all $i$. Suppose that there exists $1 \leq r < s$ such that $\sqrt{Q_i+Q_j} = \mathfrak{m}$ for each $r < j \leq s$ and $1 \leq i \leq r$. Then* $\operatorname{depth}(I) = 2$.

*Proof.* The proof follows by using Depth Lemma on the following exact sequence.

$$0 \longrightarrow S/I \longrightarrow S/\cap_{i=1}^{r} Q_i \oplus S/\cap_{j>r}^{s} Q_j \longrightarrow S/\cap_{i=1}^{r}\cap_{j>r}^{s}(Q_i+Q_j) \longrightarrow 0.$$

$\square$

**Corollary 1.5.** *Let $I \subset S$ be a monomial ideal. If bigsize of $I$ is one then $I$ has minimal depth.*

*Proof.* We know that $\operatorname{size}(I) \leq \operatorname{bigsize}(I)$. If $\operatorname{size}(I) = 0$ the $\operatorname{depth}(I) = 1$ and the result follows in this case. Now let us suppose that $\operatorname{size}(I) = 1$. By Lemma 1.4 we have $\operatorname{depth}(I) = 2$. Hence the result follows. $\square$

Let $d \geq 2$ be an integer and $\mathcal{M}_d$ the set of monomials of degree $d$ of $S$. For two monomials $u, v \in \mathcal{M}_d$, with $u \geq_{lex} v$, we consider the lexsegment set

$$\mathcal{L}(u,v) = \{w \in \mathcal{M}_d | u \geq_{lex} w \geq_{lex} v\}.$$

**Theorem 1.6.** *Let $I = (\mathcal{L}(u,v)) \subset S$ be a lexsegment ideal. Then* $\operatorname{depth}(I) = \operatorname{size}(I)+1$, *that is $I$ has minimal depth.*

*Proof.* For the trivial cases $u = v$ the result is obvious. Suppose that $u = x_1^{a_1}\cdots x_n^{a_n}$,
$v = x_1^{b_1}\cdots x_n^{b_n} \in S$. First assume that $b_1 = 0$. If there exist $r$ such that $a_1 = \cdots = a_r = 0$ and $a_{r+1} \neq 0$, then $I$ is a lexsegment ideal in $S' := K[x_{r+1}, \ldots, x_n]$. We get $\operatorname{depth}(IS) = \operatorname{depth}(IS') + r$ and by definition of size we have $\operatorname{size}(IS) = \operatorname{size}(IS') + r$. This means that without loss of generality we can assume that $a_1 > 0$. If $x_n u/x_1 \geq_{lex} v$, then by [3, Proposition 3.2] $\operatorname{depth}(I) = 1$ which implies that $\mathfrak{m} \in \operatorname{Ass}(S/I)$, thus $\operatorname{size}(I) = 0$ and the result

follows in this case. Now consider the complementary case $x_nu/x_1 <_{lex} v$, then $u$ is of the form $u = x_1x_l^{a_l}\cdots x_n^{a_n}$ where $l \geq 2$. Let $I = \cap_{i=1}^s Q_i$ be an irredundant primary decomposition of $I$, where $Q_i's$ are monomial primary ideals. If $l \geq 4$ and $v = x_2^d$ then by [3, Proposition 3.4] we have $\operatorname{depth}(I) = l-1$. After [6, Proposition 2.5($ii$)] we know that

$$\sqrt{\sum_{i=1}^s Q_i} = (x_1, x_2, x_l, \dots, x_n) \notin \operatorname{Ass}(S/I),$$

but $(x_1, x_2), (x_2, x_l, \dots, x_n) \in \operatorname{Ass}(S/I)$. Therefore, $\operatorname{size}(I) = l-2$ and we have $\operatorname{depth}(I) = \operatorname{size}(I) + 1$, so we are done in this case. Now consider the case $v = x_2^{d-1}x_j$ for some $3 \leq j \leq n-2$ and $l \geq j+2$, then again by [3, Proposition 3.4] we have $\operatorname{depth}(I) = l-j+1$ and by [6, Proposition 2.5($ii$)] we have

$$\sqrt{\sum_{i=1}^s Q_i} = (x_1, \dots, x_j, x_l, \dots, x_n) \notin \operatorname{Ass}(S/I)$$

and $(x_1, \dots, x_j), (x_2, \dots, x_j, x_l, \dots, x_n) \in \operatorname{Ass}(S/I)$. Therefore, $\operatorname{size}(I) = l-j$ and again we have $\operatorname{depth}(I) = \operatorname{size}(I) + 1$. Now for all the remaining cases by [3, Proposition 3.4] we have $\operatorname{depth}(I) = 2$, and by [6, Proposition 2.5($i$)]

$$\sqrt{\sum_{i=1}^s Q_i} = (x_1, \dots, x_n) \notin \operatorname{Ass}(S/I),$$

but $(x_1, \dots, x_j), (x_2, \dots, x_n) \in \operatorname{Ass}(S/I)$, for some $j \geq 2$. Therefore $\operatorname{size}(I) = 1$. Thus the equality $\operatorname{depth}(I) = \operatorname{size}(I) + 1$ follows in all cases when $b_1 = 0$.

Now let us consider that $b_1 > 0$, then $I = x_1^{b_1}I'$ where $I' = (I : x_1^{b_1})$. Clearly $I'$ is a lexsegment ideal generated by the lexsegment set $\mathcal{L}(u', v')$ where $u' = u/x_1^{b_1}$ and $v' = v/x_1^{b_1}$. The ideals $I$, $I'$ are isomorphic, therefore $\operatorname{depth}(I') = \operatorname{depth}(I)$. It is enough to show that $\operatorname{size}(I') = \operatorname{size}(I)$. We have the exact sequence

$$0 \to S/I' \xrightarrow{x_1^{b_1}} S/I \to S/(I, x_1^{b_1}) = S/(x_1^{b_1}) \to 0,$$

and therefore

$$\operatorname{Ass}(S/I') \subset \operatorname{Ass}(S/I) \subset \operatorname{Ass}(S/I') \cup \{(x_1)\}.$$

As $\{(x_1)\} \in \operatorname{Ass}(S/I)$ since it is a minimal prime over $I$, we get $\operatorname{Ass}(S/I) = \operatorname{Ass}(S/I') \cup \{(x_1)\}$. Let $s'$ be the minimum number such that there exist $P_1, \dots, P_s \in \operatorname{Ass} S/I'$ such that $\sum_{i=1}^s P_i = a := \sum_{P \in \operatorname{Ass}(S/I')} P$. Then size$(I') = s' + \dim(S/a) - 1$. Let $s$ be the minimum number $t$ such that there exist $t$ prime ideals in $\mathrm{Ass}(S/I)$ whose sum is $(a, x_1)$. By [6, Lemma 2.1] we have that atleast one prime ideal from $\mathrm{Ass}(S/I')$ contains necessarily $x_1$, we have $x_1 \in a$. It follows $s \leq s'$ because anyway $\sum_{i=1}^{s'} P_i = a = \sum_{P \in \mathrm{Ass}(S/I)} P$. If we have $P'_1, \ldots, P'_{s-1} \in \mathrm{Ass}(S/I')$ such that $\sum_{i=1}^{s-1} P'_i + (x_1) = a$ then we have also $\sum_{i=1}^{s-1} P'_i + P_1 = a$ for some $P_1 \in \mathrm{Ass}(S/I')$ which contains $x_1$. Thus $s = s'$ and so size$(I) =$ size$(I')$. □

## 2 Stanley depth of cyclic modules defined by ideals of minimal depth

Using Corollaries 1.3, 1.5 and Theorems 1.6, 0.2 we get the following theorem.

**Theorem 2.1.** *Stanley's conjecture holds for $I$, if it satisfies one of the following statements:*

1. $P_i \nsubseteq \sum_{1=j\neq i}^{s} P_j$ *for all* $i \in [s]$,
2. *the bigsize of $I$ is one,*
3. *$I$ is a lexsegment ideal.*

**Remark 2.2.** Usually, if Stanley's conjecture holds for an ideal $I$ then we may show that it holds for the module $S/I$ too. There exist no general explanation for this fact. If $I$ is a monomial ideal of bigsize one then Stanley's conjecture holds for $S/I$. Indeed, case $\mathrm{depth}(S/I) = 0$ is trivial. Suppose $\mathrm{depth}(S/I) \neq 0$, then by Lemma 1.4 $\mathrm{depth}(S/I) = 1$, therefore by [2, Proposition 2.13] $\mathrm{sdepth}(S/I) \geq 1$. If $I$ is a lexsegment ideal then Stanley's conjecture holds for $S/I$ [6]. Below we show this fact in the first case of the above theorem.

**Theorem 2.3.** *Let* $I = \bigcap_{i=1}^{s} Q_i$ *be the irredundant presentation of $I$ as an intersection of primary monomial ideals. Let* $P_i := \sqrt{Q_i}$. *If* $P_i \not\subset \sum_{1=i\neq j}^{s} P_j$ *for all* $i \in [s]$ *then* $\mathrm{sdepth}(S/I) \geq \mathrm{depth}(S/I)$, *that is the Stanley's conjecture holds for $S/I$.*

*Proof.* Using [4, Lemma 3.6] it is enough to consider the case $\sum_{i=1}^{s} P_i = \mathfrak{m}$. By Proposition 1.2 we have $\mathrm{depth}(S/I) = s - 1$. We show that $\mathrm{sdepth}(S/I) \geq s - 1$. Apply induction on $s$, case $s = 1$ being clear. Fix $s > 1$ and apply induction on $n$. If $n \leq 5$ then the result follows by [10]. Let $A := \cup_{i=1}^{s}(G(P_i) \setminus \sum_{1=j\neq i}^{s} G(P_j))$. If $(A) = \mathfrak{m}$ then note that $G(P_i) \cap G(P_j) = \emptyset$ for all $i \neq j$. By [7, Theorem 2.1] and [8, Theorem 3.1] we have $\mathrm{sdepth}(S/I) \geq s - 1$. Now suppose that $(A) \neq \mathfrak{m}$. By renumbering the primes and variables we can assume that $x_n \notin A$. There exists a number $r$, $2 \leq r \leq s$ such that $x_n \in G(P_j)$, $1 \leq j \leq r$ and $x_n \notin G(P_j)$, $r+1 \leq j \leq s$. Let $S' := K[x_1, \ldots, x_{n-1}]$. First assume that $r < s$. Let $Q'_j = Q_j \cap S'$, $P'_j = P_j \cap S'$ and $J = \bigcap_{i=r+1}^{s} Q'_i \subset S'$, $L = \bigcap_{i=1}^{r} Q'_i \subset S'$. We have $(I, x_n) = ((J \cap L), x_n)$ because $(Q_j, x_n) = (Q'_j, x_n)$ using the structure of monomial primary ideals given in [15]. In the exact sequence

$$0 \longrightarrow S/(I : x_n) \longrightarrow S/I \longrightarrow S/(I, x_n) \longrightarrow 0,$$

the sdepth of the right end is $\geq s-1$ by induction hypothesis on $n$ for $J \cap L \subset S'$ (note that we have $P'_i \not\subset \sum_{1=i\neq j}^{s} P'_j$ for all $i \in [s]$ since $x_n \notin A$). Let $e_I$ be the maximum degree in $x_n$ of a monomial from $G(I)$. Apply induction on $e_I$. If $e_I = 1$ then $(I : x_n) = JS$ and the sdepth of the left end in the above exact sequence is equal with $\operatorname{sdepth}(S/JS) \geq (s-r-1)+r = s-1$ since there are at least $r$ variables which do not divide the minimal monomial generators of ideal $(I : x_n)$ and we may apply induction hypothesis on $s$ for $J$. By [13, Theorem 3.1] we have $\operatorname{sdepth}(S/I) \geq \min\{\operatorname{sdepth}(S/(I : x_n)), \operatorname{sdepth}(S/(I, x_n))\} \geq s-1$. If $e_I > 1$ then note that $e_{(I:x_n)} < e_I$ and by induction hypothesis on $e_I$ or $s$ we get $\operatorname{sdepth}(S/(I : x_n)) \geq s-1$. As above we obtain by [13, Theorem 3.1] $\operatorname{sdepth}(S/I) \geq s-1$.

Now let $r = s$. If $e_I = 1$ then $I = (L, x_n)$ and by induction on $n$ we have $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(S'/L) \geq s-1$. If $e_I > 1$ then by induction hypothesis on $e_I$ and $s$ we get $\operatorname{sdepth}(S/(I : x_n)) \geq s-1$. As above we are done using [13, Theorem 3.1]. $\square$

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