



Inexact Inertial Proximal Algorithm for Maximal Monotone Operators

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Abstract

In this paper, convergence of the sequence generated by the inexact form of the inertial proximal algorithm is studied. This algorithm which is obtained by the discretization of a nonlinear oscillator with damping dynamical system, has been introduced by Alvarez and Attouch (2001) and Jules and Maingé (2002) for the approximation of a zero of a maximal monotone operator. We establish weak and strong convergence results for the inexact inertial proximal algorithm with and without the summability assumption on errors, under different conditions on parameters. Our theorems extend the results on the inertial proximal algorithm established by Alvarez and Attouch (2001) and Jules and Maingé (2002) as well as the results on the standard proximal point algorithm established by Brézis and Lions (1978), Lions (1978), Djafari Rouhani and Khatibzadeh (2008) and Khatibzadeh (2012). We also answer questions of Alvarez and Attouch (2001).

1 Introduction and preliminaries

Let H be a real Hilbert space with inner product (\cdot, \cdot) and norm $|\cdot|$. We denote weak convergence in H by \rightharpoonup and strong convergence by \rightarrow . Let A be a nonempty subset of $H \times H$ to which we shall refer as a (nonlinear) possibly multivalued operator in H . A is called monotone (resp. strongly

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monotone) iff $(y_2 - y_1, x_2 - x_1) \geq 0$ (resp. $(y_2 - y_1, x_2 - x_1) \geq \alpha|x_2 - x_1|^2$ for some $\alpha > 0$), for all $[x_i, y_i] \in A$, $i = 1, 2$. A is maximal monotone if A is monotone and $R(I + A) = H$, where I is the identity operator on H . Given any function $\varphi : H \rightarrow]-\infty, +\infty]$ (not necessarily convex) with the domain $D(\varphi)$, its subdifferential is the multivalued operator $\partial\varphi$, defined as

$$\partial\varphi(x) := \{w \in H \mid \varphi(x) - \varphi(y) \leq (w, x - y), \quad \forall y \in H\}.$$

The function φ is called proper iff there exists $x \in H$ such that $\varphi(x) < +\infty$. It is a well-known result that if φ is a proper, convex, and lower semicontinuous function, then $\partial\varphi$ is a maximal monotone operator. We refer the reader to the book by Morosanu [15] in order to understand monotone operators and subdifferential of convex functions in Hilbert spaces.

The most popular algorithm for approximation of a zero of a maximal monotone operator is the following algorithm

$$\begin{cases} u_{n+1} = J_{\lambda_{n+1}}(u_n + e_n), & n \geq 0 \\ u_0 \in H, \end{cases} \quad (1.1)$$

where $J_\lambda = (I + \lambda A)^{-1}$, A is a maximal monotone operator, $\{\lambda_n\}$ is a positive real sequence and $\{e_n\}$ is a suitable sequence in H . This iterative scheme is called the proximal point algorithm and was introduced by Martinet in [14] with $\lambda_n \equiv \lambda$ and then generalized by Rockafellar [16]. The iterative method (1.1) is also a discrete version of the following first order nonhomogeneous evolution equation of maximal monotone type

$$\begin{cases} u'(t) + Au(t) \ni f(t) \\ u(0) = u_0 \in \overline{D(A)}. \end{cases} \quad (1.2)$$

In [1-5,9] the authors considered the following iterative method for the approximation of a zero of a maximal monotone operator A .

$$\begin{cases} u_{n+1} = J_{\lambda_n}(u_n + \alpha_n(u_n - u_{n-1})) \\ u_0, u_1 \in H, \end{cases} \quad (1.3)$$

where α_n (resp. λ_n) is nonnegative (resp. positive) sequence. They obtained some results on the convergence of u_n to a zero of the operator A under suitable assumptions on λ_n and α_n . Obviously, the algorithm (1.1) is a special case of (1.3) when $\alpha_n \equiv 0$. Jules and Maingé in [9] showed that the iterative method (1.3) provide a better rate of convergence to a zero of A , than the standard proximal point algorithm. The iterative method (1.3) is called inertial

proximal algorithm and can be considered as a discrete version of nonlinear oscillator with damping dynamical system

$$\begin{cases} u''(t) + \gamma u'(t) + Au(t) \ni 0, \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \quad (1.4)$$

where A is a general maximal monotone operator and γ is a real positive number. In [2] Alvarez and Attouch proposed two open questions. The first one was related to the convergence of the inexact version of (1.3) and the other one was to develop a general theory to guide the choice of the parameters λ_k and α_k .

In this paper, we consider the inexact version of the inertial proximal algorithm

$$\begin{cases} u_{n+1} = J_{\lambda_{n+1}}(u_n + \alpha_n(u_n - u_{n-1}) + e_n), \\ u_0, u_1 \in H, \end{cases} \quad (1.5)$$

where α_n (resp. λ_n) is nonnegative (resp. positive) sequence, $\{e_n\}$ is an error sequence in H and A is a maximal monotone operator. Weak and strong convergence results are proved, under different conditions on parameters and errors. Our theorems extend the results of [2, 6, 7, 9, 11, 12].

In Section 2 of this paper, we investigate the relation between the boundedness of the sequence $\{u_n\}$ generated by (1.5) and the assumption $A^{-1}(0) \neq \emptyset$. Also the weak convergence of the weighted average of the sequence $\{u_n\}$ is proved. In Section 3, we assume that $\{u_n\}$ is bounded and show that the set of all weak cluster points of the sequence $\{u_n\}$ is a subset of $A^{-1}(0)$. Then we prove the weak convergence theorem for the sequence $\{u_n\}$ that is one of the main results of the paper and extends the weak convergence theorem of Attouch and Alvarez [2]. Section 4 is devoted to the special but important case $A = \partial\varphi$, where φ is a proper, convex and lower semicontinuous function. We prove the weak convergence of the sequence $\{u_n\}$ generated by (1.5) when $A = \partial\varphi$ with different conditions on parameters and error. In Section 5, the strong convergence of the sequence is studied when the maximal monotone operator A is strongly monotone or the resolvent of A is a compact operator.

We recall some notations, definitions and lemmas that we need in the sequel.

Definition 1.1. Given a bounded sequence $\{u_n\}$ in H , the asymptotic center c of $\{u_n\}$ is defined as follows (see [8]): for every $q \in H$, let $\varphi(q) = \lim_{n \rightarrow \infty} \sup |u_n - q|^2$. Then φ is a continuous and strictly convex function on H , satisfying $\varphi(q) \rightarrow \infty$ as $|q| \rightarrow \infty$. Thus φ achieves its minimum on H at a unique point c , called the asymptotic center of the sequence $\{u_n\}$.

Lemma 1.2. [11] Suppose that $\{\alpha_n\}$ is a nonnegative sequence and $\{\lambda_n\}$ is a positive sequence such that $\sum_{n=1}^{+\infty} \lambda_n = +\infty$. If $\frac{\alpha_n}{\lambda_n} \rightarrow 0$ as $n \rightarrow +\infty$, then $\frac{\sum_{k=1}^n \alpha_k}{\sum_{k=1}^n \lambda_k} \rightarrow 0$ as $n \rightarrow +\infty$.

Lemma 1.3. [13] Let $\{\phi_n\} \subset [0, +\infty[$ and $\{\delta_n\} \subset [0, +\infty[$ satisfy

(a) $\phi_{n+1} - \phi_n \leq \theta_n(\phi_n - \phi_{n-1}) + \delta_n$,

(b) $\sum_n \delta_n < +\infty$,

(c) $\{\theta_n\} \subset [0, \theta[$, where $\theta \in [0, 1[$.

Then $\{\phi_n\}$ is a converging sequence and $\sum_n [\phi_{n+1} - \phi_n]_+ < \infty$, where $[t]_+ := \max\{t, 0\} \forall t \in \mathbb{R}$.

Lemma 1.4. Suppose that a_n and b_n are nonnegative real sequences and that $\sum_{n=1}^{+\infty} b_n < +\infty$. If $a_{n+1} \leq a_n + b_n$, for all $n \geq 1$, then there exists $\lim_n a_n$.

Notation 1.5. Throughout the paper φ is a proper, convex and lower semi-continuous function, A is a maximal monotone operator and $w_n := (\sum_{i=1}^n \lambda_{i+1})^{-1} \sum_{i=1}^n \lambda_{i+1} u_{i+1}$ is the weighted average of the sequence $\{u_n\}$. We denote by $\omega_w(x_n)$ the set of all weak cluster points of the sequence x_n , and by Au_{n+1} the element $\frac{u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) + e_n}{\lambda_{n+1}}$ in H .

2 Boundedness of the Sequence and an Ergodic Theorem

In this section we first study the relation between boundedness of the sequence $\{u_n\}$ and the assumption $A^{-1}(0) \neq \emptyset$ as well as the existence of $\lim |u_n - p|$ for each $p \in A^{-1}(0)$. Then an ergodic theorem for the weak convergence of the weighted average of the sequence $\{u_n\}$ is studied.

Proposition 2.1. Suppose $A^{-1}(0) \neq \emptyset$ and $\{u_n\}$ is given by (1.5).

(a) If $e_n \equiv 0$ and

$$\begin{cases} (\alpha_1) \exists \alpha \in [0, 1[\text{ such that } \forall k \in \mathbb{N}, 0 \leq \alpha_k \leq \alpha, \\ (\alpha_2) \sum_{n=1}^{+\infty} \alpha_n |u_n - u_{n-1}|^2 < +\infty, \end{cases} \quad (2.1)$$

then $\lim_n |u_n - p|$ exists and $\sum_{n=1}^{\infty} [|u_n - p| - |u_{n-1} - p|]_+ < \infty$, $\forall p \in A^{-1}(0)$.

(b) If $\{u_n\}$ is bounded, $(E_1) \sum_{n=1}^{+\infty} |e_n| < +\infty$ and (α_1) and (α_2) are satisfied, then for all $p \in A^{-1}(0)$, $\lim_n |u_n - p|$ exists and $\sum_{n=1}^{\infty} [|u_n - p| - |u_{n-1} - p|]_+ < \infty$.

(c) If $(E_1) \sum_{n=1}^{+\infty} |e_n| < +\infty$ and $\sum_{n=1}^{+\infty} \alpha_n |u_n - u_{n-1}| < +\infty$, then for all $p \in A^{-1}(0)$, $\lim_n |u_n - p|$ exists.

Proof. In order to prove (a), see [2, Theorem 2.1]. Let us prove (b). Suppose that $p \in A^{-1}(0)$, then by (1.5) and the monotonicity of A

$$\begin{aligned} & 0 \geq (u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}) - e_n, u_{n+1} - p) \\ & = (u_{n+1} - u_n, u_{n+1} - p) - \alpha_n(u_n - u_{n-1}, u_{n+1} - u_n) - \alpha_n(u_n - u_{n-1}, u_n - p) - \\ & \quad (e_n, u_{n+1} - p) \\ & = \frac{1}{2}|u_{n+1} - p|^2 - \frac{1}{2}|u_n - p|^2 - \frac{\alpha_n^2}{2}|u_n - u_{n-1}|^2 + \frac{1}{2}|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})|^2 - \\ & \quad \frac{\alpha_n}{2}|u_n - u_{n-1}|^2 - \frac{\alpha_n}{2}|u_n - p|^2 + \frac{\alpha_n}{2}|u_{n-1} - p|^2 - (e_n, u_{n+1} - p). \end{aligned}$$

Set $\phi_n = \frac{1}{2}|u_n - p|^2$. We have

$$\phi_{n+1} - \phi_n - \alpha_n(\phi_n - \phi_{n-1}) \leq \alpha_n|u_n - u_{n-1}|^2 + |e_n||u_{n+1} - p|.$$

Hence by Lemma 1.3, (α_2) and (E_1) , there exists $\lim_n |u_n - p|$ and

$$\sum_{n=1}^{\infty} [|u_n - p| - |u_{n-1} - p|]_+ < \infty.$$

Now we prove (c). By nonexpansivity of the resolvent operator and (1.5), we have

$$|u_{n+1} - p| \leq |u_n - p| + \alpha_n|u_n - u_{n-1}| + e_n,$$

thus by Lemma 1.4, (α_3) and (E_1) , $\lim_n |u_n - p|$ exists. \square

In the following proposition, under the suitable conditions, we prove that the boundedness of the sequence $\{u_n\}$ generated by (1.5) implies $A^{-1}(0) \neq \emptyset$. Also, we show that the set of weak cluster points of the weighted average of the sequence u_n , say w_n , is a subset of $A^{-1}(0)$.

Proposition 2.2. *Let $\{u_n\}$ be a bounded sequence generated by (1.5). If the following conditions hold:*

$$\begin{cases} (\Lambda_1) \sum_{n=1}^{+\infty} \lambda_n = +\infty, \\ (\alpha_3) \sum_{n=1}^{+\infty} \alpha_n|u_n - u_{n-1}| < +\infty \text{ or } (\alpha_4) \frac{\alpha_n|u_n - u_{n-1}|}{\lambda_{n+1}} \rightarrow 0, \\ (E_1) \sum_{n=1}^{+\infty} |e_n| < +\infty \text{ or } (E_2) \frac{|e_n|}{\lambda_{n+1}} \rightarrow 0, \end{cases} \quad (2.2)$$

then $A^{-1}(0) \neq \emptyset$ and $\omega_w(w_n) \subset A^{-1}(0)$, where $\omega_w(w_n)$ is the set of weak cluster points of w_n .

Proof. Suppose $[x, y] \in A$, since $\{u_n\}$ is bounded, there is a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that $w_{n_j} \rightharpoonup p \in H$. On the other hand, by monotonicity of A , we get

$$\begin{aligned} (x - w_{n_j}, y) & = (x - (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} u_{i+1}, y) \\ & = (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} (x - u_{i+1}, y) \\ & = (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} ((x - u_{i+1}, y - Au_{i+1}) + (x - u_{i+1}, Au_{i+1})) \\ & \geq (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} (x - u_{i+1}, \lambda_{i+1} Au_{i+1}) \\ & = (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} (x - u_{i+1}, u_i - u_{i+1} + \alpha_i(u_i - u_{i-1}) + e_i) \\ & = (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} (x - u_{i+1}, (1 + \alpha_i)(u_i - x) - \alpha_i(u_{i-1} - x) + x - u_{i+1} + e_i) \\ & = (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} \lambda_{i+1} [- (1 + \alpha_i)(u_{i+1} - x, u_i - x) + \alpha_i(u_{i+1} - x, u_{i-1} - \end{aligned}$$

$$\begin{aligned}
 & x) + |u_{i+1} - x|^2 + (x - u_{i+1}, e_i)] \\
 &= (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} [-(u_{i+1} - x, u_i - x) + \alpha_i(u_{i+1} - x, u_{i-1} - u_i) + \\
 & |u_{i+1} - x|^2 + (x - u_{i+1}, e_i)] \\
 &\geq (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} [\frac{1}{2}|u_{i+1} - x|^2 - \frac{1}{2}|u_i - x|^2 + \frac{1}{2}|u_{i+1} - u_i|^2 - \\
 & \alpha_i|u_{i+1} - x||u_i - u_{i-1}| - |e_i||u_{i+1} - x|] \\
 &\geq (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} \sum_{i=0}^{n_j-1} [\frac{1}{2}(|u_{i+1} - x|^2 - |u_i - x|^2) - \alpha_i|u_{i+1} - x||u_i - u_{i-1}| - |e_i||u_{i+1} - x|] \\
 &\geq (\sum_{i=0}^{n_j-1} \lambda_{i+1})^{-1} [-\frac{1}{2}|u_0 - x|^2 - \sum_{i=0}^{n_j-1} \alpha_i|u_{i+1} - x||u_i - u_{i-1}| - \sum_{i=0}^{n_j-1} |e_i||u_{i+1} - x|].
 \end{aligned}$$

Letting $j \rightarrow \infty$, by Lemma 1.1 and (2.2), we get $(x - p, y) \geq 0$. Thus by maximality of A , we have $p \in A^{-1}(0)$, as desired. \square

Remark 2.3. If $e_n \equiv 0$ in (1.5) and the conditions $(\alpha_1), (\alpha_2), [(\alpha_3) \text{ or } (\alpha_4)]$ and (Λ_1) are satisfied, then by Proposition 2.2 and part (a) of Proposition 2.1, $\{u_n\}$ is bounded if and only if $A^{-1}(0) \neq \emptyset$.

Theorem 2.4. Let $\{u_n\}$ be a bounded sequence generated by (1.5). Suppose that the conditions $(\alpha_1), (\alpha_2), [(\alpha_3) \text{ or } (\alpha_4)]$ and (Λ_1) are satisfied. If $(E_1) \sum_{n=1}^{+\infty} |e_n| < +\infty$, then $w_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow \infty$, which is also the asymptotic center of $\{u_n\}$.

Proof. By Proposition 2.2, $A^{-1}(0) \neq \emptyset$ and $\omega_w(w_n) \subset A^{-1}(0)$, thus by part (b) of Proposition 2.1, there exists $\lim_n |u_n - p|$, $\forall p \in \omega_w(w_n)$. We show that $\omega_w(w_n)$ is singleton. Suppose $p, q \in \omega_w(w_n)$ and $p \neq q$, then there exists $\lim_n (|u_n - p|^2 - |u_n - q|^2)$, hence $\lim_{n \rightarrow +\infty} (u_n, p - q)$ exists. This follows that $\lim_{n \rightarrow +\infty} (w_n, p - q)$ exists. This implies that $(q, p - q) = (p, p - q)$ and hence $p = q$. So, $w_n \rightarrow p \in A^{-1}(0)$ as $n \rightarrow +\infty$. Now, we show that p is the asymptotic center of $\{u_n\}$. Suppose that $q \in H$ and $q \neq p$, then

$$|u_n - p|^2 = |u_n - q|^2 + 2(u_n, q - p) + |p|^2 - |q|^2.$$

Multiplying both sides of the above equality by λ_n , summing up from $n = 1$ to $n = m$ and dividing by $\sum_{n=1}^m \lambda_n$. Taking limsup as $m \rightarrow +\infty$, we get

$$\lim_{n \rightarrow +\infty} |u_n - p|^2 = \limsup_{m \rightarrow +\infty} (\sum_{n=1}^m \lambda_n)^{-1} (\sum_{n=1}^m \lambda_n |u_n - q|^2) - |q - p|^2 < \limsup_{n \rightarrow +\infty} |u_n - q|^2.$$

It shows that p is the asymptotic center of the sequence u_n as desired. \square

Remark 2.5. By Remark 2.1, if $e_n \equiv 0$, we can replace the boundedness of u_n by $A^{-1}(0) \neq \emptyset$ in Theorem 2.4.

3 Weak Convergence of the Algorithm with Errors

The main result of this section is to prove weak convergence of the sequence $\{u_n\}$ generated by (1.5) to a zero of the maximal monotone operator A provided that the suitable assumptions on the parameters λ_n and α_n and appropriate assumptions on the error sequence $\{e_n\}$.

Lemma 3.1. *Suppose that $\{u_n\}$ is the bounded sequence generated by (1.5). If the conditions*

$$(\alpha_5) \sum_{n=1}^{+\infty} \frac{\alpha_n^2 |u_n - u_{n-1}|^2}{\lambda_{n+1}^2} < +\infty \quad \text{and} \quad (E_3) \sum_{n=1}^{+\infty} \frac{|e_n|^2}{\lambda_{n+1}^2} < +\infty$$

are satisfied, then there exists $\lim_{n \rightarrow +\infty} |Au_n|$.

Proof. By monotonicity of A , we have

$$(Au_{n+1} - Au_n, u_{n+1} - u_n) \geq 0.$$

The equation (1.5) implies that

$$(Au_{n+1} - Au_n, \alpha_n(u_n - u_{n-1}) + e_n - \lambda_{n+1}Au_{n+1}) \geq 0.$$

Then

$$\begin{aligned} |Au_{n+1}|^2 &\leq (Au_n, Au_{n+1}) + (Au_{n+1} - Au_n, \frac{\alpha_n}{\lambda_{n+1}}(u_n - u_{n-1}) + \frac{e_n}{\lambda_{n+1}}) \\ &\leq \frac{1}{2}|Au_n|^2 + \frac{1}{2}|Au_{n+1}|^2 - \frac{1}{2}|Au_n - Au_{n+1}|^2 + \frac{1}{2}|Au_n - Au_{n+1}|^2 \\ &\quad + \frac{1}{2}|\frac{\alpha_n}{\lambda_{n+1}}(u_n - u_{n-1}) + \frac{e_n}{\lambda_{n+1}}|^2 \\ &\leq \frac{1}{2}|Au_n|^2 + \frac{1}{2}|Au_{n+1}|^2 + \frac{\alpha_n^2}{\lambda_{n+1}^2}|u_n - u_{n-1}|^2 + \frac{|e_n|^2}{\lambda_{n+1}^2}. \end{aligned}$$

Therefore

$$|Au_{n+1}|^2 \leq |Au_n|^2 + 2\frac{\alpha_n^2}{\lambda_{n+1}^2}|u_n - u_{n-1}|^2 + 2\frac{|e_n|^2}{\lambda_{n+1}^2}. \quad (3.1)$$

Then there exists $\lim_{n \rightarrow +\infty} |Au_n|$, by (α_5) , (E_3) and Lemma 1.4. \square

Proposition 3.2. *Let $\{u_n\}$ be a bounded sequence generated by (1.5). If (α_5) , (E_3) , and the following conditions hold:*

$$\begin{cases} (\Lambda_2) \sum_{n=1}^{+\infty} \lambda_n^2 = +\infty, \\ (\alpha_3) \sum_{n=1}^{+\infty} \alpha_n |u_n - u_{n-1}| < +\infty \quad \text{or} \quad (\alpha_6) \frac{\alpha_n |u_n - u_{n-1}|}{\lambda_{n+1}^2} \rightarrow 0, \\ (E_1) \sum_{n=1}^{+\infty} |e_n| < +\infty \quad \text{or} \quad (E_4) \frac{|e_n|}{\lambda_{n+1}^2} \rightarrow 0, \end{cases} \quad (3.2)$$

then $\lim_{n \rightarrow +\infty} |Au_n| = 0$, $A^{-1}(0) \neq \emptyset$ and $\omega_w(u_n) \subset A^{-1}(0)$.

Proof. Set $L = \sup_{n \geq 1} |Au_n|$. By Proposition 2.2, $A^{-1}(0) \neq \emptyset$ (because $(\Lambda_2) \Rightarrow (\Lambda_1)$, $(\alpha_5) \Rightarrow (\alpha_4)$ and $(E_3) \Rightarrow (E_2)$). Assume $p \in A^{-1}(0)$. By monotonicity of A

$$(\lambda_{n+1}Au_{n+1}, u_{n+1} - p) \geq 0.$$

By (1.5), we get

$$(u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p) \geq 0,$$

which implies

$$(\alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p) \geq (u_{n+1} - u_n, u_{n+1} - p).$$

Thus, by (1.5)

$$(\alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p) \geq (\alpha_n(u_n - u_{n-1}) + e_n - \lambda_{n+1}Au_{n+1}, u_{n+1} - p),$$

hence

$$\begin{aligned} & 2\alpha_n|u_n - u_{n-1}||u_{n+1} - p| + 2|e_n||u_{n+1} - p| + |u_n - p|^2 - |u_{n+1} - p|^2 \\ & \geq |\alpha_n(u_n - u_{n-1}) + e_n - \lambda_{n+1}Au_{n+1}|^2 = \lambda_{n+1}^2 \left| \frac{\alpha_n}{\lambda_{n+1}}(u_n - u_{n-1}) + \frac{e_n}{\lambda_{n+1}} - Au_{n+1} \right|^2 \\ & = \lambda_{n+1}^2 \left[\left| \frac{\alpha_n}{\lambda_{n+1}}(u_n - u_{n-1}) + \frac{e_n}{\lambda_{n+1}} \right|^2 + |Au_{n+1}|^2 - 2 \left(\frac{\alpha_n}{\lambda_{n+1}}(u_n - u_{n-1}) + \frac{e_n}{\lambda_{n+1}}, Au_{n+1} \right) \right] \\ & \geq \lambda_{n+1}^2 \left[|Au_{n+1}|^2 - 2 \left| \frac{\alpha_n}{\lambda_{n+1}}(u_n - u_{n-1}) + \frac{e_n}{\lambda_{n+1}} \right| |Au_{n+1}| \right] \\ & \geq \lambda_{n+1}^2 \left[|Au_{n+1}|^2 - 2 \frac{\alpha_n}{\lambda_{n+1}} |u_n - u_{n-1}| |Au_{n+1}| - 2 \frac{|e_n|}{\lambda_{n+1}} |Au_{n+1}| \right]. \end{aligned}$$

So

$$\begin{aligned} \lambda_{n+1}^2 |Au_{n+1}|^2 & \leq 2L\lambda_{n+1}\alpha_n|u_n - u_{n-1}| + 2L\lambda_{n+1}|e_n| + 2\alpha_n|u_n - u_{n-1}||u_{n+1} - p| \\ & \quad + 2|e_n||u_{n+1} - p| + |u_n - p|^2 - |u_{n+1} - p|^2. \end{aligned} \quad (3.3)$$

Summing up both sides of (3.3) from $n = 1$ to $n = k$ and then dividing by $\sum_{n=1}^k \lambda_{n+1}^2$. $|Au_n| \rightarrow l$ as $n \rightarrow +\infty$ (by Lemma 3.1) implies that $(\sum_{n=1}^k \lambda_{n+1}^2)^{-1} \sum_{n=1}^k \lambda_{n+1}^2 |Au_{n+1}|^2 \rightarrow l$ as $k \rightarrow +\infty$, then by assumptions and Lemma 1.2, we get $l = 0$. Now, if $u_{n_j} \rightharpoonup q$, then by demicloseness of A , we get $q \in A^{-1}(0)$, hence $\omega_w(u_n) \subset A^{-1}(0)$. \square

Remark 3.3. Proposition 3.2 shows that if $A^{-1}(0)$ is a singleton (which happens if, for example, A is strictly monotone), then $u_n \rightarrow p$, where p is the unique element of $A^{-1}(0)$.

Remark 3.4. Although Proposition 3.2 does not imply the weak convergence of u_n to $p \in A^{-1}(0)$ unless $A^{-1}(0)$ are singleton, it improves the errors in the inertial proximal algorithm. It shows that the error sequence $\{|e_n|\}$ can go to infinity, provided that the suitable assumptions on λ_n and α_n hold. For example, $|e_n| = n$, $\alpha_n = \frac{1}{n^2}$ and $\lambda_n = (n-1)^2$ satisfy the assumptions of Proposition 3.2 for any bounded sequence $\{u_n\}$.

In the following theorem, we obtain the weak convergence of the sequence $\{u_n\}$ generated by (1.5). Our result extends the result of [2, Theorem 2.1].

Theorem 3.5. *Let $\{u_n\}$ be a bounded sequence generated by (1.5). Suppose that the conditions (Λ_2) , (α_1) , (α_2) , (α_5) , (E_1) and (E_3) are satisfied. Then $u_n \rightharpoonup p \in A^{-1}(0)$ as $n \rightarrow \infty$.*

Proof. By Proposition 2.2, $A^{-1}(0) \neq \emptyset$. Assume $p \in A^{-1}(0)$. By monotonicity of A , we have

$$(u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}) - e_n, u_{n+1} - p) \leq 0.$$

Thus

$$\begin{aligned} & \lambda_{n+1}^2 |Au_{n+1}|^2 + |u_{n+1} - p|^2 \leq |u_n - p + \alpha_n(u_n - u_{n-1}) + e_n|^2 \\ & \leq |u_n - p|^2 + \alpha_n^2 |u_n - u_{n-1}|^2 + |e_n|^2 + 2\alpha_n(u_n - p, u_n - u_{n-1}) + \\ & 2\alpha_n(e_n, u_n - u_{n-1}) + 2(u_n - p, e_n) \\ & \leq |u_n - p|^2 + \alpha_n^2 |u_n - u_{n-1}|^2 + |e_n|^2 + \alpha_n |u_n - p|^2 + \alpha_n |u_n - u_{n-1}|^2 - \\ & \alpha_n |u_{n-1} - p|^2 + \alpha_n |u_n - u_{n-1}|^2 + \alpha_n |e_n|^2 + 2|e_n||u_n - p|. \end{aligned}$$

Therefore

$$\begin{aligned} & \lambda_{n+1}^2 |Au_{n+1}|^2 \leq |u_n - p|^2 - |u_{n+1} - p|^2 + 3\alpha_n |u_n - u_{n-1}|^2 + \\ & \alpha_n (|u_n - p|^2 - |u_{n-1} - p|^2) + 2|e_n|^2 + 2|e_n||u_n - p| \\ & \leq |u_n - p|^2 - |u_{n+1} - p|^2 + 3\alpha_n |u_n - u_{n-1}|^2 + \\ & \alpha[|u_n - p|^2 - |u_{n-1} - p|^2]_+ + 2|e_n|^2 + 2|e_n||u_n - p|. \end{aligned}$$

Summing up from $n = 1$ to k , we get

$$\sum_{n=1}^k \lambda_{n+1}^2 |Au_{n+1}|^2 \leq \sum_{n=1}^k (|u_n - p|^2 - |u_{n+1} - p|^2) + 3 \sum_{n=1}^k \alpha_n |u_n - u_{n-1}|^2 + \alpha \sum_{n=1}^k [|u_n - p|^2 - |u_{n-1} - p|^2]_+ + 2 \sum_{n=1}^k |e_n|^2 + 2 \sum_{n=1}^k |e_n||u_n - p|.$$

Letting $k \rightarrow \infty$, by part (b) of Proposition 2.1 and the assumptions, we obtain

$$\sum_{n=1}^{\infty} \lambda_{n+1}^2 |Au_{n+1}|^2 < +\infty, \quad (3.4)$$

which implies $\liminf_n |Au_n| = 0$. Since by Lemma 3.1, $\lim_n |Au_n|$ exists, thus $\lim_n |Au_n| = 0$. Now if $u_{n_j} \rightharpoonup q$, then by demiclosedness of A , we have $q \in A^{-1}(0)$, hence $\omega_w(u_n) \subset A^{-1}(0)$. By a similar proof of that of Theorem 2.4, we can show $\omega_w(u_n)$ is singleton. Therefore $u_n \rightharpoonup p \in A^{-1}(0)$. \square

Remark 3.6. Theorem 3.5 in exact form ($e_n \equiv 0$) extends Theorem 2.1 of [2]. Because if $e_n \equiv 0$, $A^{-1}(0) \neq \emptyset$ and $\lambda_n \geq \lambda > 0$, then (Λ_2) is satisfied and

$$\sum_{n=1}^{\infty} \frac{\alpha_n^2 |u_n - u_{n-1}|^2}{\lambda_{n+1}^2} \leq \frac{\alpha}{\lambda^2} \sum_{n=1}^{\infty} \alpha_n |u_n - u_{n-1}|^2 < +\infty,$$

by (α_1) and (α_2) . Then (α_5) is also satisfied.

4 Subdifferential Case

In this section, we establish the weak convergence of the sequence $\{u_n\}$ generated by (1.5) to an element of $A^{-1}(0)$, when $A = \partial\varphi$.

Lemma 4.1. *Suppose that $\{u_n\}$ is the bounded sequence generated by (1.5) and $A = \partial\varphi$. If the conditions*

$$(\alpha_7) \sum_{n=1}^{+\infty} \frac{\alpha_n^2 |u_n - u_{n-1}|^2}{\lambda_{n+1}} < +\infty \quad \text{and} \quad (E_5) \sum_{n=1}^{+\infty} \frac{|e_n|^2}{\lambda_{n+1}} < +\infty.$$

are satisfied, then there exists $\lim_{n \rightarrow +\infty} |Au_n|$.

Proof. By subdifferential inequality and (1.5), we get
 $\lambda_{n+1}(\varphi(u_{n+1}) - \varphi(u_n)) \leq (\lambda_{n+1}\partial\varphi(u_{n+1}), u_{n+1} - u_n)$
 $= (u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - u_n)$
 $= -|u_{n+1} - u_n|^2 + (\alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - u_n)$
 $\leq \frac{1}{2}|\alpha_n(u_n - u_{n-1}) + e_n|^2 \leq \alpha_n^2|u_n - u_{n-1}|^2 + |e_n|^2.$
 Hence

$$\varphi(u_{n+1}) \leq \varphi(u_n) + \frac{\alpha_n^2}{\lambda_{n+1}}|u_n - u_{n-1}|^2 + \frac{|e_n|^2}{\lambda_{n+1}}. \quad (4.1)$$

By assumptions (α_7) , (E_5) and Lemma 1.4, there exists $\lim_{n \rightarrow +\infty} \varphi(u_n)$. \square

The following proposition is the subdifferential case of Proposition 3.2.

Proposition 4.2. *Let $\{u_n\}$ be a bounded sequence generated by (1.5) and $A = \partial\varphi$. If (α_7) , (E_5) , and the following conditions hold:*

$$\begin{cases} (\Lambda_1) \sum_{n=1}^{+\infty} \lambda_n = +\infty, \\ (\alpha_3) \sum_{n=1}^{+\infty} \alpha_n |u_n - u_{n-1}| < +\infty \quad \text{or} \quad (\alpha_4) \frac{\alpha_n |u_n - u_{n-1}|}{\lambda_{n+1}} \rightarrow 0, \\ (E_1) \sum_{n=1}^{+\infty} |e_n| < +\infty \quad \text{or} \quad (E_2) \frac{|e_n|}{\lambda_{n+1}} \rightarrow 0, \end{cases} \quad (4.2)$$

then $\lim \varphi(u_n) = \inf_{x \in H} \varphi$, $(\partial\varphi)^{-1}(0) \neq \emptyset$ and $\omega_w(u_n) \subset (\partial\varphi)^{-1}(0)$.

Proof. By Proposition 2.2, $A^{-1}(0) \neq \emptyset$. Assume that $p \in (\partial\varphi)^{-1}(0)$, we have
 $\lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) \leq (\lambda_{n+1}\partial\varphi(u_{n+1}), u_{n+1} - p)$
 $= (u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p)$
 $= (u_n - p, u_{n+1} - p) + (p - u_{n+1}, u_{n+1} - p) + (\alpha_n(u_n - u_{n-1}), u_{n+1} - p) + (e_n, u_{n+1} - p).$

So,

$$\lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) \leq \frac{1}{2}|u_n - p|^2 - \frac{1}{2}|u_{n+1} - p|^2 +$$

$$\alpha_n |u_n - u_{n-1}| |u_{n+1} - p| + |e_n| |u_{n+1} - p|. \quad (4.3)$$

Summing up both sides of (4.3) from $n = 1$ to $n = k$ and dividing by $\sum_{n=1}^k \lambda_{n+1}$. Letting $k \rightarrow \infty$, since by Lemma 4.1, $\lim_{n \rightarrow +\infty} \varphi(u_n) - \varphi(p) = l$ exists, then

$$\lim_{k \rightarrow +\infty} \left(\sum_{n=1}^k \lambda_{n+1} \right)^{-1} \sum_{n=1}^k \lambda_{n+1} (\varphi(u_{n+1}) - \varphi(p)) = \lim_{k \rightarrow +\infty} (\varphi(u_n) - \varphi(p)) = l.$$

Now the assumptions and Lemma 1.2 implies that $l = 0$. Thus $\lim_k \varphi(u_k) = \varphi(p)$. Therefore, if $u_{n_j} \rightarrow q$, then $\varphi(q) \leq \liminf_j \varphi(u_{n_j}) = \varphi(p)$, which implies $q \in (\partial\varphi)^{-1}(0)$. Hence $\omega_w(u_n) \subset (\partial\varphi)^{-1}(0)$. \square

The following theorem shows that in the special case $A = \partial\varphi$, which is important from optimization point of view, the conditions (Λ_2) and (α_5) in Proposition 3.2 can be replaced by the weaker conditions (Λ_1) and (α_4) .

Theorem 4.3. *Let $\{u_n\}$ be a bounded sequence generated by (1.5) and $A = \partial\varphi$. Assume that the conditions (Λ_1) , (α_1) , (α_2) , (α_7) , (E_1) and $(E_5) \sum_{n=1}^{+\infty} \frac{|e_n|^2}{\lambda_{n+1}} < +\infty$ are satisfied. Then $u_n \rightarrow p \in (\partial\varphi)^{-1}(0)$ as $n \rightarrow \infty$.*

Proof. By Proposition 2.2, $A^{-1}(0) \neq \emptyset$. Assume $p \in (\partial\varphi)^{-1}(0)$, we have

$$\begin{aligned} \lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) &\leq (\lambda_{n+1} \partial\varphi(u_{n+1}), u_{n+1} - p) \\ &= (u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p) \\ &= (p - u_{n+1}, u_{n+1} - p) + (u_n - p + \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p) \\ &\leq -\frac{1}{2}|u_{n+1} - p|^2 + \frac{1}{2}|u_n - p + \alpha_n(u_n - u_{n-1}) + e_n|^2 \\ &= -\frac{1}{2}|u_{n+1} - p|^2 + \frac{1}{2}|u_n - p|^2 + \frac{\alpha_n^2}{2}|u_n - u_{n-1}|^2 + \frac{1}{2}|e_n|^2 \\ &\quad + (u_n - p, e_n) + \alpha_n(u_n - u_{n-1}, u_n - p) + (\alpha_n(u_n - u_{n-1}), e_n) \\ &\leq \frac{1}{2}|u_n - p|^2 - \frac{1}{2}|u_{n+1} - p|^2 + 3\frac{\alpha_n}{2}|u_n - u_{n-1}|^2 + \frac{\alpha_n}{2}|u_n - p|^2 - \frac{\alpha_n}{2}|u_{n-1} - p|^2 + |e_n|^2 + |e_n||u_n - p|. \end{aligned}$$

So

$$\begin{cases} \lambda_{n+1}(\varphi(u_{n+1}) - \varphi(p)) \leq \frac{1}{2}|u_n - p|^2 - \frac{1}{2}|u_{n+1} - p|^2 + 3\frac{\alpha_n}{2}|u_n - u_{n-1}|^2 \\ + \frac{\alpha}{2}[|u_n - p|^2 - |u_{n-1} - p|^2]_+ + |e_n|^2 + |e_n||u_n - p|. \end{cases} \quad (4.4)$$

On the other hand, by Lemma 4.1, there exists $\lim_{n \rightarrow +\infty} (\phi(u_n) - \phi(p))$. Let $\lim_{n \rightarrow +\infty} (\phi(u_n) - \phi(p)) = l$. Multiplying both sides of (4.4) by λ_{n+1} , summing up from $n = 1$ to $n = k$ and then dividing by $\sum_{n=1}^k \lambda_{n+1}$. Letting $k \rightarrow +\infty$ then

$$\lim_{k \rightarrow +\infty} \left(\sum_{n=1}^k \lambda_{n+1} \right)^{-1} \sum_{n=1}^k \lambda_{n+1} (\phi(u_n) - \phi(p)) = l.$$

Thus by assumptions on $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{e_n\}$ and Lemma 1.2, we get: $l = 0$. Therefore $\lim_{n \rightarrow +\infty} \phi(u_n) = \phi(p)$. Now if $u_{n_j} \rightharpoonup q$, then $\varphi(q) \leq \liminf_j \varphi(u_{n_j}) = \varphi(p)$ implies $q \in (\partial\varphi)^{-1}(0)$. Hence $\omega_w(u_n) \subset (\partial\varphi)^{-1}(0)$. The rest of the proof is similar to that of Theorem 2.4. \square

Remark 4.4. Obviously $(\Lambda_2) \Rightarrow (\Lambda_1)$, on the other hand by (α_1) , (α_2) and (α_5) , we have:

$$\begin{aligned} \sum_{n=1}^{+\infty} \frac{\alpha_n^2 |u_n - u_{n-1}|^2}{\lambda_{n+1}} &\leq \left(\sum_{n=1}^{+\infty} \frac{\alpha_n^2 |u_n - u_{n-1}|^2}{\lambda_{n+1}^2} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{+\infty} \alpha_n^2 |u_n - u_{n-1}|^2 \right)^{\frac{1}{2}} \\ &\leq \alpha^{\frac{1}{2}} \left(\sum_{n=1}^{+\infty} \frac{\alpha_n^2 |u_n - u_{n-1}|^2}{\lambda_{n+1}^2} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{+\infty} \alpha_n |u_n - u_{n-1}|^2 \right)^{\frac{1}{2}} < +\infty. \end{aligned}$$

Then (α_7) is satisfied. This shows that, in subdifferential case and when $e_n \equiv 0$, the weak convergence of u_n is proved under the weaker assumptions on the parameters.

5 Strong Convergence

In this final section, the strong convergence of the bounded sequence $\{u_n\}$ generated by (1.5) to a zero of A is studied under additional assumptions on the maximal monotone operator A .

Theorem 5.1. *Assume that $\{u_n\}$ is a bounded sequence generated by (1.5). If $(I + A)^{-1}$ is a compact operator and the conditions (Λ_2) , (α_1) , (α_2) and (E_1) are satisfied, then $u_n \rightarrow p \in A^{-1}(0)$.*

Proof. By (3.4) and the assumptions, we get: $\liminf_n |Au_n| = 0$. Therefore there exists a subsequence $\{Au_{n_j}\}$ of $\{Au_n\}$ such that $|Au_{n_j}| \rightarrow 0$ and $\{u_{n_j} + Au_{n_j}\}$ is bounded. Since $(I + A)^{-1}$ is compact, $\{u_{n_j}\}$ has a strongly convergent subsequence (we denote again by $\{u_{n_j}\}$) to $p \in H$. The maximality of A implies that $p \in A^{-1}(0)$. On the other hand, by part (b) of Proposition 2.1, $\lim_n |u_n - p|$ exists. Hence $u_n \rightarrow p \in A^{-1}(0)$. \square

Lemma 5.2. [11] *Assume $\{y_n\}$ is a positive real sequence satisfying the following inequality:*

$$b_n y_n \leq y_{n-1} - y_n + a_n,$$

where $\{b_n\}$ and $\{a_n\}$ are positive sequences, then we have:

- (i) *If $\{\frac{a_n}{b_n}\}$ is bounded, then the sequence $\{y_n\}$ is bounded.*
- (ii) *If $\lim_n \frac{a_n}{b_n} = 0$, then there exists $\lim_n y_n$.*
- (iii) *If $\lim_n \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{+\infty} b_n = +\infty$, then $\lim_n y_n = 0$.*

Theorem 5.3. *Let $\{u_n\}$ be a bounded sequence generated by (1.5) and A be a maximal monotone and strongly monotone operator. If (A_1) , (α_4) and (E_2) are satisfied, then $u_n \rightarrow p$, where p is the unique element of $A^{-1}(0)$.*

Proof. By Proposition 2.2, $A^{-1}(0) \neq \emptyset$. Assume that p is the single element of $A^{-1}(0)$. By the strong monotonicity of A and (1.5), we get

$$\alpha\lambda_{n+1}|u_{n+1} - p|^2 \leq (u_n - u_{n+1} + \alpha_n(u_n - u_{n-1}) + e_n, u_{n+1} - p).$$

It follows that

$$2\alpha\lambda_{n+1}|u_{n+1} - p|^2 \leq |u_n - p|^2 - |u_{n+1} - p|^2 + 2\alpha_n|u_n - u_{n-1}||u_{n+1} - p| + 2|e_n||u_{n+1} - p|.$$

The theorem follows by the assumptions and Lemma 5.2. \square

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