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Weighted differentiation composition operators from the logarithmic Bloch space to the weighted-type space

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Abstract

The boundedness of the weighted differentiation composition operator from the logarithmic Bloch space to the weighted-type space is characterized in terms of the sequence $(z^n)_{n\in\mathbb{N}_0}$. An asymptotic estimate of the essential norm of the operator is also given in terms of the sequence, which gives a characterization for the compactness of the operator.

1 Introduction

Let X and Y be two Banach spaces. A linear operator $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. The essential norm of an operator $T : X \to Y$ is its distance to the space of compact operators, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} :\ K : X \to Y \quad \text{is compact}\},$$

where $\|\cdot\|_{X\to Y}$ is the operator norm. It is easy to see that $\|T\|_{e,X\to Y} = 0$ if and only if T is compact.

Let $H(\mathbb{D})$ be the class of all holomorphic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane. Recently, there has been a great interest in studying product-type operators between spaces of holomorphic functions on

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the unit disk or the open unit ball in the n-dimensional complex vector space $\mathbb{C}^n$ (see, e.g., [6], [8]-[22], [25], [27], [29]-[50], [53], [55]-[57] and the related references therein).

The differentiation operator D on $H(\mathbb{D})$ is defined by $Df = f'$, $f \in H(\mathbb{D})$. For a nonnegative integer n, we define

$$(D^0 f)(z) = f(z), \quad (D^n f)(z) = f^{(n)}(z),$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

Let $u \in H(\mathbb{D})$. The multiplication operator on $H(\mathbb{D})$, denoted by M_u, is defined by

$$(M_u f)(z) = u(z) f(z),$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator on $H(\mathbb{D})$, denoted by C_φ, is defined by

$$(C_\varphi f)(z) = f(\varphi(z)),$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

These three operators are some of the basic ones and are involved in the definition of the operator studied in this paper.

Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$ and let n be a nonnegative integer. The *weighted differentiation composition operator* or *generalized weighted composition operator*, which was probably introduced for the first time in [55] and is usually denoted by $D^n_{\varphi,u}$, is the product-type operator defined as follows

$$D^n_{\varphi,u}(f)(z) = u(z) \cdot (D^n f)(\varphi(z)),$$

where $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. Note that the operator can be written in the following product-type form $D^n_{\varphi,u} = M_u \circ C_\varphi \circ D^n$.

When $n = 0$ and $u(z) = 1$, $D^n_{\varphi,u}$ is the composition operator C_φ. When $n = 0$, $D^n_{\varphi,u}$ is the weighted composition operator uC_φ, which is the following product of the composition operator and the multiplication operator $M_u \circ C_\varphi$. Both operators are studied a lot (see, e.g., [4, 5, 6, 23, 24, 51, 52] and the references therein). For $n = 1$ and $u \equiv 1$ or $u = \varphi'$ are obtained products of composition and differentiation operators which are studied, for example, in [13, 14, 17, 18, 25, 33, 35, 38, 43]. Operator $D^n_{\varphi,u}$ and some of its special cases, were studied, for example, in [11, 19, 20, 37, 41, 42, 46, 53, 55, 56, 57]. For some other related product-type operators including, among others, composition and differentiation operators, see, e.g., [8, 9, 21, 22, 45, 47, 48].

Let us say, that beside this class of product-type operators, the classes including integral-type operators (see, e.g., [3, 28]) also attracted some attention (see, e.g., [10, 15, 16, 27, 29, 30, 31, 32, 34, 36, 39, 40, 44, 49, 50] and the related references therein). These integral-type operators include indirectly multiplication and differentiation operators too. For example, the operators introduced in [15] and [16] acting on the spaces of holomorphic functions on the unit disc are of this sort. For the case of the open unit ball see the operator in [32] (it includes the radial differentiation operator, which is more suitable for dealing with holomorphic functions of several variables). Some of the integral-type operators does not contain the differentiation, but only the multiplication one (see, e.g., [10, 27, 30, 34, 36, 40]).

A basic problem concerning all these operators on various spaces of holomorphic functions is to relate their operator theoretic properties to the function theoretic properties of the involving symbols. For some applications of methods of functional analysis on various spaces of functions and related topics, see, e.g., [5, 26].

Now we present the spaces on which will be considered the operator studied in the paper.

The logarithmic-Bloch space, denoted by $\mathcal{LB}$, is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\log} = \sup_{z\in\mathbb{D}}(1-|z|)\left(\ln \frac{e}{1-|z|}\right)|f'(z)| < \infty.$$

$\mathcal{LB}$ is a Banach space with the norm $\|f\|_{\mathcal{LB}} = |f(0)| + \|f\|_{\log}$. From [1] we see that $\mathcal{LB}\bigcap H^\infty$ is the space of multipliers of the Bloch space $\mathcal{B}$. Here the Bloch space is defined as follows

$$\mathcal{B} = \Big\{f \in H(\mathbb{D}) : \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty\Big\}.$$

As usual, a positive continuous function on $\mathbb{D}$ is called *weight*. Each weight $\mu = \mu(z)$ on $\mathbb{D}$ defines the *weighted-type space*, as follows (see, e.g., [2, 6])

$$H^\infty_\mu = H^\infty_\mu(\mathbb{D}) = \{f \in H(\mathbb{D}) : \|f\|_{H^\infty_\mu} < \infty\},$$

where

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{D}} \mu(z)|f(z)|$$

is a norm on the space.

Studying the boundedness, compactness and essential norm of the composition operator on the Bloch space attracted considerable attention in the last few decades see, e.g., [23, 24, 51, 52]. For example, in [51] it was proved that the composition operator acting on the Bloch space is compact if and only if

$$\lim_{j\to\infty} \|C_\varphi(z^j)\|_{\mathcal{B}} = 0.$$

Motivated by [51], Colonna and Li characterized the boundedness and compactness of the operator $uC_\varphi : H^\infty \to \mathcal{LB}$ in [4]. Among other results, they proved that $uC_\varphi : H^\infty \to \mathcal{LB}$ is bounded if and only if

$$\sup_{j\in\mathbb{N}_0} \|uC_\varphi(z^j)\|_{\mathcal{LB}} < \infty.$$

In [19] the authors of this paper characterized the boundedness and compactness of the operator $D^n_{\varphi,u}$ from α-Bloch spaces (for the definition of the space see, e.g., [33, 54]) into weighted-type spaces in a similar way. For some other results on essential norm of concrete operators, see, e.g., [5, 6, 19, 33, 48, 53].

Here, we investigate the boundendness, compactness and give an estimate for the essential norm of the operator $D^n_{\varphi,n} : \mathcal{LB} \to H^\infty_\mu$ in terms of the sequence $(\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu})^\infty_{j=n}$. This paper is a continuation of the above mentioned line of investigations. We would also like to mention that there has been some interest in studying logarithimic-type spaces and operators from or to them (see, e.g., [4, 7, 10, 21, 29, 31, 34, 39, 40, 53]).

Recall that, two real sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are *asymptotically equivalent* if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, and we write $a_n \sim b_n$. We say that $P \preceq Q$ if there exists a constant C such that $P \le CQ$. The symbol $P \approx Q$ means that $P \preceq Q \preceq P$.

2 The boundedness of $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$

In this section, we state and prove a boundedness criterion for the operator $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$. For this purpose, we first quote some auxiliary results which will be used in the proofs of the main results in this paper. The following technical lemma was proved in [53].

Lemma 1. *For $n, j \in \mathbb{N}$, define the function $G_{n,j} : [0,1) \to [0,\infty)$ by*

$$G_{n,j}(x) = \frac{j!}{(j-n)!} x^{j-n}(1-x)^n \ln\frac{e}{1-x}.$$

Then the following statements hold.

(i) For $j \ge n$, there is a unique $x_{n,j} \in [0,1)$ such that $G_{n,j}(x_{n,j})$ is the absolute maximum of $G_{n,j}$.

(ii) $\lim_{j\to\infty} x_{n,j} = 1$, $\lim_{j\to\infty}[j(1-x_{n,j})] = n$ *and*

$$\lim_{j\to\infty} \frac{\max_{0<t<1} G_{n,j}(t)}{\ln(j+1)} = \left(\frac{n}{e}\right)^n.$$

(iii) For $j - n > 0$, *let* $r_{n,j} = (j-n)/j$. *Then* $G_{n,j}$ *is increasing on* $[r_{n,j-n}, r_{n,j}]$ *and*

$$\min_{r_{n,j-n}\le x\le r_{n,j}} G_{n,j}(x) = G_{n,j}(r_{n,j-n}) \sim \left(\frac{n}{e}\right)^n \ln(j+1), \quad \text{as} \quad j\to\infty.$$

Moreover,

$$\min_{r_{n,j-n}\le x\le r_{n,j}} \frac{G_{n,j}(x)}{\|z^j\|_{\mathcal{LB}}} = \frac{G_{n,j}(r_{n,j-n})}{\|z^j\|_{\mathcal{LB}}} \sim \frac{n^n}{e^{n-1}}, \quad \text{as} \quad j\to\infty.$$

Remark 1. Note that in the last asymptotic relation was used the fact that

$$\|z^j\|_{\mathcal{LB}} \sim \frac{\ln(j+1)}{e},$$

which follows from Lemma 1 *(ii)* with $n = 1$.

The following folklore lemma, can be found, for example, in [7].

Lemma 2. *Let* $m \in \mathbb{N}$. *Then* $f \in \mathcal{LB}$ *if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|)^m\left(\ln\frac{e}{1-|z|}\right)|f^{(m)}(z)| < \infty.$$

Moreover,

$$\|f\|_{\mathcal{LB}} \asymp \sum_{j=0}^{m-1}|f^{(j)}(0)| + \sup_{z\in\mathbb{D}}(1-|z|)^m\left(\ln\frac{e}{1-|z|}\right)|f^{(m)}(z)|. \tag{1}$$

The main result in this section is the following.

Theorem 1. *Let* $n \in \mathbb{N}$, μ *be a weight,* $u \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Then* $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ *is bounded if and only if*

$$M := \sup_{j\in\mathbb{N}_0} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}} < \infty. \tag{2}$$

Proof. First, we assume that (2) holds. Then for $j = n$, we get $u \in H_\mu^\infty$. Assume $\|\varphi\|_\infty := \sup_{z\in\mathbb{D}} |\varphi(z)| < 1$. By (1) it follows that there is a positive constant C_n such that

$$\sup_{z\in\mathbb{D}}(1-|z|)^n\left(\ln\frac{e}{1-|z|}\right)|f^{(n)}(z)| \le C_n\|f\|_{\mathcal{LB}}, \tag{3}$$

for every $f \in H(\mathbb{D})$.

From (3) and the monotonicity of the function $g_n(x) = x^n \ln(e/x)$ on the interval $(0,1]$ (for a closely related statement see, e.g., [10, Lemma 1]), we have

$$\begin{aligned}\|D^n_{\varphi,u}(f)\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{D}} \mu(z)|u(z)f^{(n)}(\varphi(z))| \\ &= \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)||f^{(n)}(\varphi(z))|(1-|\varphi(z)|)^n \ln\frac{e}{1-|\varphi(z)|}}{(1-|\varphi(z)|)^n\ln\frac{e}{1-|\varphi(z)|}} \\ &\le \frac{C_n\|u\|_{H^\infty_\mu}\|f\|_{\mathcal{LB}}}{(1-\|\varphi\|_\infty)^n\ln\frac{e}{1-\|\varphi\|_\infty}} < \infty, \end{aligned}\tag{4}$$

for any $f \in \mathcal{LB}$.

On the other hand, we have

$$n!\|u\|_{H^\infty_\mu} = \|D^n_{\varphi,u}(z^n)\|_{H^\infty_\mu} = \|z^n\|_{\mathcal{LB}}\frac{\|D^n_{\varphi,u}(z^n)\|_{H^\infty_\mu}}{\|z^n\|_{\mathcal{LB}}} \le \|z^n\|_{\mathcal{LB}}M. \tag{5}$$

Hence, from (4) and (5) it follows that the operator $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded in this case, and moreover

$$\|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \le \frac{\widehat{C}_n M}{(1-\|\varphi\|_\infty)^n\ln\frac{e}{1-\|\varphi\|_\infty}}, \tag{6}$$

where constant $\widehat{C}_n = C_n\|z^n\|_{\mathcal{LB}}/n!$ depends on n only.

Now assume that $\|\varphi\|_\infty = 1$. Let $N \ge 2n+1$ be the smallest positive integer such that $\mathbb{D}_N$ is not empty, where $\mathbb{D}_j = \{z \in \mathbb{D} : r_{n,j-n} \le |\varphi(z)| \le r_{n,j}\}$ and $r_{n,j}$ is given in Lemma 1. Note that $G_{n,j}(|\varphi(z)|) > 0$, when $z \in \mathbb{D}_j, j \ge N$, so by Lemma 1 we obtain

$$\delta_\varphi := \inf_{j\ge N}\inf_{z\in\mathbb{D}_j}\frac{G_{n,j}(|\varphi(z)|)}{\|z^j\|_{\mathcal{LB}}} > 0.$$

We have

$$\begin{aligned}\|D^n_{\varphi,u}(f)\|_{H^\infty_\mu} &= \sup_{z\in\mathbb{D}}\mu(z)|u(z)f^{(n)}(\varphi(z))| \\ &= \max\left\{\sup_{j\ge N}\sup_{z\in\mathbb{D}_j}\mu(z)|u(z)f^{(n)}(\varphi(z))|, \sup_{z\in\mathbb{D}_{N-1}}\mu(z)|u(z)f^{(n)}(\varphi(z))|\right\}. \end{aligned}\tag{7}$$

By Lemma 2 and (3), we have that for any given $f \in \mathcal{LB}$

$$\begin{aligned}
&\sup_{j\geq N}\sup_{z\in\mathbb{D}_j}\mu(z)|u(z)f^{(n)}(\varphi(z))| \\
&= \sup_{j\geq N}\sup_{z\in\mathbb{D}_j}\mu(z)|u(z)f^{(n)}(\varphi(z))|\frac{\|z^j\|_{\mathcal{LB}}}{G_{n,j}(|\varphi(z)|)}\frac{G_{n,j}(|\varphi(z)|)}{\|z^j\|_{\mathcal{LB}}} \\
&\leq C_n\frac{\|f\|_{\mathcal{LB}}}{\delta_\varphi}\sup_{j\geq N}\sup_{z\in\mathbb{D}_j}\frac{j!}{(j-n)!}\mu(z)|u(z)|\frac{|\varphi(z)|^{j-n}}{\|z^j\|_{\mathcal{LB}}} \\
&\leq C_n\frac{\|f\|_{\mathcal{LB}}}{\delta_\varphi}\sup_{j\geq N}\frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}}. \qquad (8)
\end{aligned}$$

On the other hand, if $N > 2n+1$ we have that $D_{N-1} = \emptyset$, so that

$$\sup_{z\in\mathbb{D}_{N-1}}\mu(z)|u(z)f^{(n)}(\varphi(z))| = 0. \qquad (9)$$

From (7), (8) and (9), it follows that, in this case, $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded, and moreover

$$\|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \leq \frac{C_n M}{\delta_\varphi}. \qquad (10)$$

If $N = 2n+1$, then $D_{N-1} = D_{2n} = \{z : |\varphi(z)| < 1/2\}$, so by (3), the monotonicity of the function $g_n(x) = x^n\ln(e/x)$ on the interval $(0,1]$ and (5), we get

$$\begin{aligned}
\sup_{z\in\mathbb{D}_{N-1}}\mu(z)|u(z)f^{(n)}(\varphi(z))| &\leq \|u\|_{H^\infty_\mu}\sup_{z\in\mathbb{D}_{2n}}\frac{|f^{(n)}(\varphi(z))|(1-|\varphi(z)|)^n\ln\frac{e}{1-|\varphi(z)|}}{(1-|\varphi(z)|)^n\ln\frac{e}{1-|\varphi(z)|}} \\
&\leq \frac{2^n C_n\|u\|_{H^\infty_\mu}\|f\|_{\mathcal{LB}}}{\ln(2e)} \\
&\leq \frac{2^n C_n\|z^n\|_{\mathcal{LB}}M}{n!\ln(2e)}\|f\|_{\mathcal{LB}}, \qquad (11)
\end{aligned}$$

for any $f \in \mathcal{LB}$.

From (7), (8) and (11), it follows that $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded in this case, and moreover

$$\|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \leq \max\left\{\frac{C_n M}{\delta_\varphi}, \frac{2^n C_n\|z^n\|_{\mathcal{LB}}M}{n!\ln(2e)}\right\}. \qquad (12)$$

Conversely, assume that $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded, i.e., $\|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} <$

∞. Since the sequence $(z^j/\|z^j\|_{\mathcal{LB}})_{j\in\mathbb{N}_0}$ is bounded in $\mathcal{LB}$, we have

$$\frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}} \leq \|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \left\| \frac{z^j}{\|z^j\|_{\mathcal{LB}}} \right\|_{\mathcal{LB}} \leq \|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} < \infty, \tag{13}$$

for any $j \in \mathbb{N}_0$, from which the implication follows. $\square$

Remark 2. Note that M in (2) is, in fact, equal to

$$\sup_{j\geq n} \|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}/\|z^j\|_{\mathcal{LB}}.$$

Remark 3. Note that from (6), (10), (12) and (13) we have that the following inequalities hold

$$M \leq \|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \leq C_{\varphi,n}M,$$

where constant $C_{\varphi,n}$ depends on φ and n. Hence, for a fixed φ we have that

$$\|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \approx M.$$

Remark 4. Note also that in the case $\|\varphi\|_\infty < 1$, the boundedness of $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ implies

$$n!\|u\|_{H^\infty_\mu} = \|D^n_{\varphi,u}(z^n)\|_{H^\infty_\mu} \leq \|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu}\|z^n\|_{\mathcal{LB}},$$

from which it follows that $u \in H^\infty_\mu$ and moreover

$$\frac{n!}{\|z^n\|_{\mathcal{LB}}}\|u\|_{H^\infty_\mu} \leq \|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu}. \tag{14}$$

From (4) and (14) we get

$$\frac{n!}{\|z^n\|_{\mathcal{LB}}}\|u\|_{H^\infty_\mu} \leq \|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \leq \frac{C_n}{(1-\|\varphi\|_\infty)^n \ln\frac{e}{1-\|\varphi\|_\infty}}\|u\|_{H^\infty_\mu},$$

for a constant C_n depending only on n, which means that for a fixed φ, $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded if and only if $u \in H^\infty_\mu$, and moreover the following asymptotic relation holds

$$\|D^n_{\varphi,u}\|_{\mathcal{LB}\to H^\infty_\mu} \approx \|u\|_{H^\infty_\mu}.$$

Using Remark 1 in Theorem 1 the following corollary is obtained.

Corollary 1. *Let $n \in \mathbb{N}$, μ be a weight, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded if and only if*

$$\sup_{j\in\mathbb{N}} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\ln(j+1)} < \infty.$$

3 The essential norm of $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$

Let $K_r f(z) = f(rz)$ for $f \in \mathcal{LB}$ and $r \in (0,1)$. It is easy to see that K_r is compact on $\mathcal{LB}$ and $\|K_r\|_{\mathcal{LB}\to\mathcal{LB}} \le 1$. Denote by I the identity operator. In order to give an estimate for the essential norm of $D^n_{\varphi,u}$ from $\mathcal{LB}$ to H^∞_μ, we need the following result, which was proved in [53].

Lemma 3. *There is a sequence $(r_k)_{k\in\mathbb{N}}$, with $0 < r_k < 1$ tending to 1 as $k \to \infty$, such that the compact operators*

$$L_j = \frac{1}{j}\sum_{k=1}^{j} K_{r_k}, \quad j \in \mathbb{N},$$

on $\mathcal{LB}$ satisfy the following conditions.

(i) *For any $t \in (0,1)$,* $\lim\limits_{j\to\infty} \sup\limits_{\|f\|_{\mathcal{LB}}\le 1} \sup\limits_{|z|\le t} |((I-L_j)f)'(z)| = 0.$

(iia) $\lim\limits_{j\to\infty} \sup\limits_{\|f\|_{\mathcal{LB}}\le 1} \sup\limits_{|z|<1} |(I-L_j)f(z)| \le 1,$

(iib) $\lim\limits_{j\to\infty} \sup\limits_{\|f\|_{\mathcal{LB}}\le 1} \sup\limits_{|z|<s} |(I-L_j)f(z)| = 0,$ *for any $s \in (0,1)$.*

(iii) $\limsup\limits_{j\to\infty} \|I-L_j\|_{\mathcal{LB}\to\mathcal{LB}} \le 1.$

The next lemma is proved by using standard Schwartz's arguments (see, e.g., Proposition 3.11 in [5]).

Lemma 4. *Let $n \in \mathbb{N}$, μ be a weight, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is compact if and only if $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded and for any bounded sequence $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{LB}$ converging to zero uniformly on compact subsets of $\mathbb{D}$, $\|D^n_{\varphi,u}(f_j)\|_{H^\infty_\mu} \to 0$ as $j \to \infty$.*

The following result gives an asymptotic estimate for the essential norm of the operator $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$.

Theorem 2. *Let $n \in \mathbb{N}$, μ be a weight, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Suppose that $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded. Then*

$$\|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu} \approx \limsup_{j\to\infty} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}}. \tag{15}$$

Proof. First note that since $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded and $p_n(z) = z^n \in \mathcal{LB}$, we have that $u \in H^\infty_\mu$. We first give the upper estimate for the essential norm. Assume $\|\varphi\|_\infty < 1$. Let $(f_j)_{j\in\mathbb{N}}$ be a bounded sequence in $\mathcal{LB}$ converging to zero uniformly on compacts of $\mathbb{D}$. From the Cauchy integral formula we have that $(f_j^{(n)})_{j\in\mathbb{N}}$ converges to zero on compact subsets of $\mathbb{D}$ as $j \to \infty$. Hence, we have

$$\begin{aligned}\lim_{j\to\infty} \|D^n_{\varphi,u}(f_j)\|_{H^\infty_\mu} &= \lim_{j\to\infty} \sup_{z\in\mathbb{D}} \mu(z)|u(z) f_j^{(n)}(\varphi(z))| \\ &\le \|u\|_{H^\infty_\mu} \lim_{j\to\infty} \sup_{z\in\mathbb{D}} |f_j^{(n)}(\varphi(z))| \\ &= \|u\|_{H^\infty_\mu} \lim_{j\to\infty} \sup_{|w|\le\|\varphi\|_\infty} |f_j^{(n)}(w)| = 0.\end{aligned}$$

From this and by Lemma 4 it follows that the operator $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is compact, which implies that

$$\|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu} = 0. \tag{16}$$

On the other hand, we have that

$$\|z^j\|_{\mathcal{LB}} \ge j t^{j-1}(1-t) \ln \frac{e}{1-t}\Big|_{t=\frac{j-1}{j}} = \left(1 - \frac{1}{j}\right)^{j-1} \ln(ej) \ge \frac{1}{e} \ln(ej),$$

for $j \ge 2$, which implies that

$$\begin{aligned}\limsup_{j\to\infty} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}} &\le e \limsup_{j\to\infty} \sup_{z\in\mathbb{D}} \mu(z) \frac{j!}{(j-n)!} |u(z)||\varphi(z)|^{j-n} \\ &\le e\|u\|_{H^\infty_\mu} \lim_{j\to\infty} j^n \|\varphi\|_\infty^{j-n} = 0. \end{aligned}\tag{17}$$

From (16) and (17), we see that (15) holds in this case.

Now we assume that $\|\varphi\|_\infty = 1$. Let $(L_j)_{j\in\mathbb{N}}$ be the sequence of operators given in Lemma 3. Since L_j is compact on $\mathcal{LB}$ and $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded, then $D^n_{\varphi,u} L_j : \mathcal{LB} \to H^\infty_\mu$ is also compact.

Hence

$$\begin{aligned}\|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu} &\le \limsup_{j\to\infty}\|D^n_{\varphi,u}-D^n_{\varphi,u}L_j\|_{\mathcal{LB}\to H^\infty_\mu}\\ &=\limsup_{j\to\infty}\|D^n_{\varphi,u}(I-L_j)\|_{\mathcal{LB}\to H^\infty_\mu}\\ &=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\|D^n_{\varphi,u}(I-L_j)(f)\|_{H^\infty_\mu}\\ &=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{z\in\mathbb{D}}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|.\end{aligned}$$

For each positive integer $i\ge n$, let $\mathbb{D}_i$ be as in the proof of Theorem 1. Let $k\ge 2n$ be the smallest positive integer such that $\mathbb{D}_k\ne\emptyset$. Since $\|\varphi\|_\infty=1$, $\mathbb{D}_i$ is not empty for every integer $i\ge k$ and $\mathbb{D}=\bigcup_{i=k}^\infty\mathbb{D}_i$.

Since, by Lemma 1, $\lim_{i\to\infty}\frac{\|z^i\|_{\mathcal{LB}}}{G_{n,i}(r_{n,i-n})}=\frac{e^{n-1}}{n^n}$, we have that for any $\varepsilon>0$, there exists $N\ge 2n+1$ such that

$$\frac{\|z^i\|_{\mathcal{LB}}}{G_{n,i}(r_{n,i-n})}\le\frac{e^{n-1}}{n^n}+\varepsilon \tag{18}$$

when $i\ge N$.

For an $\varepsilon>0$ we find $N=N(\varepsilon)$ such that (18) holds. We have

$$\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{z\in\mathbb{D}}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|=I_1(j)+I_2(j),$$

where

$$I_1(j)=\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{k\le i\le N-1}\sup_{z\in\mathbb{D}_i}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|$$

and

$$I_2(j)=\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{i\ge N}\sup_{z\in\mathbb{D}_i}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|.$$

For such N it follows that

$$\begin{aligned}I_2(j)&=\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{i\ge N}\sup_{z\in\mathbb{D}_i}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|\\ &=\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{i\ge N}\sup_{z\in\mathbb{D}_i}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|\frac{G_{n,i}(|\varphi(z)|)}{\|z^i\|_{\mathcal{LB}}}\frac{\|z^i\|_{\mathcal{LB}}}{G_{n,i}(|\varphi(z)|)}\\ &\le C_n\Big(\frac{e^{n-1}}{n^n}+\varepsilon\Big)\sup_{\|f\|_{\mathcal{LB}}\le 1}\|(I-L_j)f\|_{\mathcal{LB}}\sup_{i\ge N}\sup_{z\in\mathbb{D}_i}\mu(z)|u(z)|\frac{i!}{(i-n)!}\frac{|\varphi(z)|^{i-n}}{\|z^i\|_{\mathcal{LB}}}\\ &\le C_n\Big(\frac{e^{n-1}}{n^n}+\varepsilon\Big)\|I-L_j\|_{\mathcal{LB}\to\mathcal{LB}}\sup_{i\ge N}\frac{\|D^n_{\varphi,u}(z^i)\|_{H^\infty_\mu}}{\|z^i\|_{\mathcal{LB}}}.\end{aligned}$$

By Lemma 3 *(iii)*, we get

$$\limsup_{j\to\infty} I_2(j) \le C_n\Big(\frac{e^{n-1}}{n^n}+\varepsilon\Big)\sup_{i\ge N}\frac{\|D^n_{\varphi,u}(z^i)\|_{H^\infty_\mu}}{\|z^i\|_{\mathcal{LB}}}.$$

By Lemma 3 *(ii)* and the Cauchy integral formula, we have

$$\begin{aligned}\limsup_{j\to\infty} I_1(j) =&\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{k\le i\le N-1}\sup_{z\in\mathbb{D}_i}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|\\ \le&\|u\|_{H^\infty_\mu}\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{|\varphi(z)|\le r_{n,N-1}}|((I-L_j)f)^{(n)}(\varphi(z))| = 0.\end{aligned}$$

Hence

$$\begin{aligned}&\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{z\in\mathbb{D}}\mu(z)|u(z)((I-L_j)f)^{(n)}(\varphi(z))|\\ =&\limsup_{j\to\infty} I_1(j)+\limsup_{j\to\infty} I_2(j)\le C_n\Big(\frac{e^{n-1}}{n^n}+\varepsilon\Big)\sup_{i\ge N}\frac{\|D^n_{\varphi,u}(z^i)\|_{H^\infty_\mu}}{\|z^i\|_{\mathcal{LB}}},\end{aligned}$$

which implies that

$$\|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu}\le C_n\Big(\frac{e^{n-1}}{n^n}+\varepsilon\Big)\sup_{i\ge N}\frac{\|D^n_{\varphi,u}(z^i)\|_{H^\infty_\mu}}{\|z^i\|_{\mathcal{LB}}}. \tag{19}$$

When $\varepsilon\to 0^+$ we have that $N\to\infty$. So letting $\varepsilon\to 0^+$ in (19), we get

$$\|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu}\preceq\limsup_{i\to\infty}\frac{\|D^n_{\varphi,u}(z^i)\|_{H^\infty_\mu}}{\|z^i\|_{\mathcal{LB}}}.$$

Now, we give the lower estimate for the essential norm of $D^n_{\varphi,u}:\mathcal{LB}\to H^\infty_\mu$. Without loss of generality, we assume that $j\ge n$. Choose the sequence of functions $f_j(z)=z^j/\|z^j\|_{\mathcal{LB}}$, $j\in\mathbb{N}$. Then $\|f_j\|_{\mathcal{LB}}=1$ and $f_j\to 0$ converges uniformly on compacts of $\mathbb{D}$, so it converges to zero weakly on $\mathcal{LB}$ as $j\to\infty$. Thus we have $\lim_{j\to\infty}\|Kf_j\|_{H^\infty_\mu}=0$ for any given compact operator $K:\mathcal{LB}\to H^\infty_\mu$. Hence

$$\|D^n_{\varphi,u}-K\|_{\mathcal{LB}\to H^\infty_\mu}\ge\|(D^n_{\varphi,u}-K)f_j\|_{H^\infty_\mu}\ge\|D^n_{\varphi,u}(f_j)\|_{H^\infty_\mu}-\|Kf_j\|_{H^\infty_\mu},$$

and consequently

$$\|D^n_{\varphi,u}-K\|_{\mathcal{LB}\to H^\infty_\mu}\ge\limsup_{j\to\infty}\|D^n_{\varphi,u}(f_j)\|_{H^\infty_\mu}.$$

Therefore, we have

$$\|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu}=\inf_K\|D^n_{\varphi,u}-K\|_{\mathcal{LB}\to H^\infty_\mu}\ge\limsup_{j\to\infty}\frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}},$$

completing the proof. □

From Theorem 2, we obtain the following two results.

Corollary 2. *Let $n \in \mathbb{N}$, μ be a weight, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the operator $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is compact if and only if $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded and*

$$\limsup_{j\to\infty} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}} = 0.$$

Corollary 3. *Let $n \in \mathbb{N}$, μ be a weight, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Suppose that $D^n_{\varphi,u} : \mathcal{LB} \to H^\infty_\mu$ is bounded. Then*

$$\|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu} \approx \limsup_{j\to\infty} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\ln(j+1)}.$$

Remark 5. From the proof of Theorem 2 we see that the following inequalities holds

$$\limsup_{j\to\infty} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}} \le \|D^n_{\varphi,u}\|_{e,\mathcal{LB}\to H^\infty_\mu} \le C_n \frac{e^{n-1}}{n^n} \limsup_{j\to\infty} \frac{\|D^n_{\varphi,u}(z^j)\|_{H^\infty_\mu}}{\|z^j\|_{\mathcal{LB}}},$$

where C_n is defined in (3).

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