DOI: 10.1515/auom-2016-0044
An. Şt. Univ. Ovidius Constanţa **Vol. 24(2),2016, 301–320**

Some properties of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra in BRK-algebras

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Abstract

The purpose of this paper is to initiate the concept of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra in BRK-algebra and investigate some of their related properties. We also show that the relationship between n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra and the crisp subalgebra in BRK-algebra are discussed.

1 Introduction

The concept of fuzzy set, which was published by Zadeh in his definitive paper [21] of 1965, was applied by many researchers to generalize some of the basic concepts of algebra. The fuzzy algebraic structures play a vital role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, real analysis, measure theory etc. In [1], Bandaru studied BRK-algebras. The notion of three dimensional fuzzy sets was introduced by Li et al. in [11]. In 2010, Shang et al. initiated the notion of n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets [20]. In [18], Shabir and Rafiq introduced the concept of n-dimensional fuzzy ideals of semigroups.

In 1971, Rosenfeld formulated the elements of theory of fuzzy groups [17]. A new type of fuzzy subgroup, which is, the $(\in, \in \vee\, q)$-fuzzy subgroup, was

Key Words: BRK-algebra; N-dimensional fuzzy subalgebra; N-dimensional (α, β)-fuzzy subalgebra; N-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra.
2010 Mathematics Subject Classification: 03G10, 03B05, 03B52, 06F35.
Received: 06.10.2014
Accepted: 07.01.2015

introduced by Bhakat and Das [4] by using the combined notions of belongingness and quasi-coincidence of fuzzy points and fuzzy sets, which was introduced by Pu and Liu [16]. Murali [15] proposed the definition of fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subsets. It was found that the most viable generalization of Rosenfeld's fuzzy subgroup is $(\in, \in \vee q)$-fuzzy subgroup. Bhakat [2, 3] introduced the concept of $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Many researchers utilized these concepts to generalize some concepts of algebra (see [5, 8, 9, 10, 11, 12, 13, 23, 24, 25, 26, 27, 28, 29]). Davvaz in [6] discussed $(\in, \in \vee q)$-fuzzy subnearrings and ideals. In [8], Jun defined the concept of (α, β)-fuzzy subalgebras in BCK/BCI-algebras, where α, β, are any of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$. Zulfiqar initiated the notion of (α, β)-fuzzy positive implicative ideals in BCK-algebras [23]. In [12], Ma et al. initiated the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals in BCI-algebras. The notion of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy soft Γ-hyperideals of Γ-hyperrings was first introduced by Zhan in [22]. Shabir and Ali [19], characterized regular semigroups by their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideals. The notions of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-interior ideals in hemirings was introduced in [14]. Zulfiqar and Shabir [29], introduced the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subcommutative ideals in BCI-algebras. In [7], Huang et al. initiated the notion of semihyperrings by their $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy hyperideals. Recently, Zulfiqar defined the concept of $(\bar{\in}_\gamma, \bar{\in}_\gamma \vee \bar{q}_\delta)$-fuzzy fantastic ideals in BCH-algebras [24].

In the present paper, we define the concept of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra in BRK-algebra and investigate some of their related properties. We also show that the relationship between n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra and the crisp subalgebra in BRK-algebra are discussed.

The definitions and terminologies that we used in this paper are standard. For notations, terminologies and applications, the readers are referred to [5, 8, 9, 10, 11, 12, 13, 23, 24, 25, 26, 27, 28, 29].
A map $\hat{\lambda} : X \to [0, 1]$ is called a n-dimensional fuzzy subset of X and denoted as

$$\hat{\lambda}(x) = (\lambda_1(x), \lambda_2(x), ..., \lambda_n(x)).$$

For more detail of n-dimensional fuzzy subset see [20].

2 Preliminaries

In what follows, let X denote a BRK-algebra unless otherwise specified.

Definition 2.1 ([1]). *A BRK-algebra X is a general algebra (X, $*$, 0) of type (2, 0) satisfying the following conditions:*
(BRK-I) $x * 0 = x$
(BRK-II) $(x * y) * x = 0 * y$
for all x, y $\in X$.

We can define a partial order $\leq$ on X by x $\leq$ y if and only if x $*$ y = 0.

Definition 2.2 ([1]). *A nonempty subset S of a BRK-algebra X is called a subalgebra of X if it satisfies* $x * y \in S$, *for all* $x, y \in S$.

We now review some fuzzy logic concepts. An n-dimensional fuzzy set $\hat{\lambda}$ of a universe X is a function from X into the unit closed interval [0, 1], that is $\hat{\lambda} : X \to [0, 1]$. For an n-dimensional fuzzy set $\hat{\lambda}$ of a BRK-algebra X and $\hat{t} \in$ (0, 1], the crisp set

$$\hat{\lambda}_{\hat{t}} = \{x \in X | \hat{\lambda}(x) \geq \hat{t}\}$$

is called the level subset of $\hat{\lambda}$.

Definition 2.3. *An n-dimensional fuzzy set* $\hat{\lambda}$ *of a BRK-algebra X is called an n-dimensional fuzzy subalgebra of X if it satisfies the condition*

$$\hat{\lambda}(x * y) \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y),$$

for all $x, y \in X$.

Theorem 2.4. *An n-dimensional fuzzy set* $\hat{\lambda}$ *of a BRK-algebras X is an n-dimensional fuzzy subalgebra of X if and only if for every* $\hat{t} \in (0, 1]$, $\hat{\lambda}_{\hat{t}} = \{x \in X | \hat{\lambda}(x) \geq \hat{t}\}$ *is a subalgebra of X.*

Proof. The proof of the following Theorem is obvious. □

An n-dimensional fuzzy set $\hat{\lambda}$ of a BRK-algebra X having the form

$$\hat{\lambda}(y) = \begin{cases} \hat{t} = (t_1, t_2, ..., t_n) \in (0, 1] & \text{if } y = x \\ \hat{0} = (0, 0, ..., 0) & \text{if } y \neq x \end{cases}$$

is said to be an n-dimensional fuzzy point with support x and value $\hat{t} = (t_1, t_2, ..., t_n)$ and is denoted by $x_{\hat{t}}$. An n-dimensional fuzzy point $x_{\hat{t}}$ is said to

belong to (resp., quasi-coincident with) a n-dimensional fuzzy set $\hat{\lambda}$, written as $x_{\hat{t}} \in \hat{\lambda}$ (resp. $x_{\hat{t}} q \hat{\lambda}$) if $\hat{\lambda}(x) \geq \hat{t}$ i.e., $\hat{\lambda}_i$(x) $\geq t_i$ for $i = 1, 2, ..., n$ (resp. $\hat{\lambda}$(x) + $\hat{t} > \hat{1}$ i.e., $\hat{\lambda}_i$(x) + $t_i > \hat{1}$ for $i = 1, 2, ..., n$). By $x_{\hat{t}} \in \vee q \hat{\lambda} (x_{\hat{t}} \in \wedge q \hat{\lambda})$ we mean that $x_{\hat{t}} \in \hat{\lambda}$ or $x_{\hat{t}} q \hat{\lambda} (x_{\hat{t}} \in \hat{\lambda}$ and $x_{\hat{t}} q \hat{\lambda})$. For all $\hat{t}_1$, $\hat{t}_2 \in$ [0, 1], min$\{\hat{t}_1, \hat{t}_2\}$ and max$\{\hat{t}_1, \hat{t}_2\}$ will be denoted by $\hat{t}_1 \wedge \hat{t}_2$ and $\hat{t}_1 \vee \hat{t}_2$, respectively.
In what follows let α and β denote any one of $\in, q, \in \vee q, \in \wedge q$ and $\alpha \neq \in \wedge q$ unless otherwise specified. To say that $x_{\hat{t}} \bar{\alpha} \hat{\lambda}$ means that $x_{\hat{t}} \alpha \hat{\lambda}$ does not hold.
If I = $(I_1, I_2, ..., I_n) \subseteq$ X, then the n-dimensional characteristic function of I is a function C_I of X onto $\{0, 1\}$ defined by:

$$C_I(x) = \begin{cases} \hat{1} = (1, 1, ..., 1) & \text{if } x \in I \\ \hat{0} = (0, 0, ..., 0) & \text{if } x \notin I \end{cases}$$

Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$, where

$$\gamma = (\gamma_1, \gamma_2, ..., \gamma_n) \text{ and } \delta = (\delta_1, \delta_2, ..., \delta_n).$$

For n-dimensional fuzzy point $x_{\hat{r}}$ and n-dimensional fuzzy set $\hat{\lambda}$ of BRK-algebra X. We defined as:
(1) $x_{\hat{r}} \in_\gamma \hat{\lambda}$ if $\hat{\lambda}(x) \geq \hat{r} > \gamma$.
(2) $x_{\hat{r}} q_\delta \hat{\lambda}$ if $\hat{\lambda}(x) + \hat{r} > 2\delta$.
(3) $x_{\hat{r}} \in_\gamma \vee q_\delta \hat{\lambda}$ if $x_{\hat{r}} \in_\gamma \hat{\lambda}$ or $x_{\hat{r}} q_\delta \hat{\lambda}$.

3 N-dimensional (α, β)-fuzzy subalgebra

In this section, we define the concept of n-dimensional (α, β)-fuzzy subalgebra in a BRK-algebra and investigate some of their properties. Throughout this paper X will denote a BRK-algebra, where α, β, are any one of $\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta$ unless otherwise specified.

Definition 3.1. *An n-dimensional fuzzy set $\hat{\lambda}$ of a BRK-algebra X is called an n-dimensional (α, β)-fuzzy subalgebra of X, where $\alpha \neq \in_\gamma \wedge q_\delta$, if it satisfies the condition*

$$x_{\hat{t}_1} \alpha \hat{\lambda},\ y_{\hat{t}_2} \alpha \hat{\lambda} \Rightarrow (x * y)_{\hat{t}_1 \wedge \hat{t}_2} \beta \hat{\lambda}$$

for all $\hat{t}_1$, $\hat{t}_2 \in (\gamma, 1]$ and x, $y \in X$.

Let $\hat{\lambda}$ be an n-dimensional fuzzy set of a BRK-algebra X such that $\hat{\lambda}(x) \leq \delta$ for all x $\in$ X. Let x $\in$ X and $\hat{t} \in (\gamma, 1]$ be such that

$$x_{\hat{t}} \in_\gamma \wedge q_\delta \hat{\lambda}.$$

Then

$$\hat{\lambda}(x) \geq \hat{t} > \gamma \text{ and } \hat{\lambda}(x) + \hat{t} > 2\delta.$$

It follows that

$$2\delta < \hat{\lambda}(x) + \hat{t} \leq \hat{\lambda}(x) + \hat{\lambda}(x) = 2\hat{\lambda}(x).$$

This implies that $\hat{\lambda}(x) > \delta$. This means that

$$\{x_{\hat{t}} | x_{\hat{t}} \in_\gamma \wedge q_\delta \hat{\lambda}\} = \phi.$$

Therefore, the case $\alpha = \in_\gamma \wedge q_\delta$ in the above definition is omitted.

Theorem 3.2. *Let $2\delta = 1 + \gamma$ and $\hat{\lambda}$ be an n-dimensional (α, β)-fuzzy subalgebra of X. Then the set*

$$\hat{\lambda}_\gamma = \{x \in X | \hat{\lambda}(x) > \gamma\}$$

is a subalgebra of X.

Proof. Let x, y $\in$ X be such that

$$\text{x} \in \hat{\lambda}_\gamma \text{ and y} \in \hat{\lambda}_\gamma.$$

Then

$$\hat{\lambda}(x) > \gamma \text{ and } \hat{\lambda}(y) > \gamma.$$

Assume that

$$\hat{\lambda}(x * y) \leq \gamma.$$

If $\alpha \in \{\in_\gamma, \in_\gamma \vee q_\delta\}$, then

$$x_{\hat{\lambda}(x)} \alpha \hat{\lambda} \text{ and } y_{\hat{\lambda}(y)} \alpha \hat{\lambda}$$

but

$$\hat{\lambda}(x * y) \leq \gamma < \hat{\lambda}(x) \wedge \hat{\lambda}(y)$$

and

$$\hat{\lambda}(x * y) + \hat{\lambda}(x) \wedge \hat{\lambda}(y) \leq \gamma + 1 = 2\delta.$$

This implies that

$$(x * y)_{\hat{\lambda}(x) \wedge \hat{\lambda}(y)} \overline{\beta} \hat{\lambda}$$

for every $\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$, which is a contradiction. Hence

$$\hat{\lambda}(x * y) > \gamma,$$

that is,

$$x * y \in \hat{\lambda}_\gamma.$$

Also

$$\hat{\lambda}(x * y) + 1 > \gamma + 1 = 2\delta.$$

This implies that $(x * y)_1 q_\delta \hat{\lambda}$, but

$$\hat{\lambda}(x * y) \leq \gamma \text{ so } (x * y)_1 \bar{\in}_\gamma \hat{\lambda}$$

and

$$\hat{\lambda}(x * y) + 1 \leq \gamma + 1 = 2\delta,$$

so

$$(x * y)_1 \bar{q}_\delta \hat{\lambda},$$

a contradiction. Hence

$$\hat{\lambda}(x * y) > \gamma,$$

that is,

$$x * y \in \hat{\lambda}_\gamma.$$

Therefore, $\hat{\lambda}_\gamma$ is a subalgebra of X. □

Theorem 3.3. *Let $2\delta = 1 + \gamma$ and I be a nonempty subset of a BRK-algebra X. Then I is a subalgebra of X if and only if the n-dimensional fuzzy subset $\hat{\lambda}$ of X defined by*

$$\hat{\lambda}(x) = \begin{cases} \geq \delta & \text{if } x \in I \\ \leq \gamma & \text{if } x \notin I \end{cases}$$

is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X.

Proof. Let I be a subalgebra of X. Let x, y $\in$ X and $\hat{t}$, $\hat{r} \in (\gamma, 1]$, be such that

$$x_{\hat{t}} \in_\gamma \hat{\lambda} \text{ and } y_{\hat{r}} \in_\gamma \hat{\lambda}.$$

Then

$$\hat{\lambda}(x) \geq \hat{t} > \gamma \text{ and } \hat{\lambda}(y) \geq \hat{t} > \gamma.$$

Thus x, y $\in$ I. Since I is a subalgebra of X, we have x $*$ y $\in$ I, that is,

$$\hat{\lambda}(x * y) \geq \delta.$$

If $\hat{t} \wedge \hat{r} \leq \delta$, then

$$\hat{\lambda}(x * y) \geq \delta \geq \hat{t} \wedge \hat{r} > \gamma,$$

which implies that

$$(x * y)_{\hat{t} \wedge \hat{r}} \in_\gamma \hat{\lambda}.$$

If $\hat{t} \wedge \hat{r} > \delta$, then

$$\hat{\lambda}(x * y) + \hat{t} \wedge \hat{r} \geq \delta + \delta = 2\delta,$$

which implies that

$$(x * y)_{\hat{t} \wedge \hat{r}} q_\delta \hat{\lambda}.$$

Thus

$$(x * y)_{\hat{t} \wedge \hat{r}} \in_\gamma \vee q_\delta \hat{\lambda}.$$

Hence $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X.

Conversely, assume that $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. Then I $= \hat{\lambda}_r$. Thus by Theorem 3.2, I is a subalgebras of X. □

Corollary 3.4. *Let $2\delta = 1 + \gamma$ and I be a nonempty subset of a BRK-algebra X. Then I is a subalgebra of X if and only if the characteristic function C_I of I is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X.*

Theorem 3.5. *Let $2\delta = 1 + \gamma$ and I be a nonempty subset of a BRK-algebra X. Then I is a subalgebra of X if and only if the n-dimensional fuzzy subset $\hat{\lambda}$ of X defined by*

$$\hat{\lambda}(x) = \begin{cases} \geq \delta & \text{if } x \in I \\ \leq \gamma & \text{if } x \notin I \end{cases}$$

is an n-dimensional $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X.

Proof. Let I be a subalgebra of X. Let x, y $\in$ X and $\hat{t}$, $\hat{r} \in (\gamma, 1]$ be such that

$$x_{\hat{t}} q_\delta \hat{\lambda} \text{ and } y_{\hat{r}} q_\delta \hat{\lambda}.$$

Then

$$\hat{\lambda}(x) + \hat{t} > 2\delta \text{ and } \hat{\lambda}(y) + \hat{t} > 2\delta,$$

which implies that

$$\hat{\lambda}(x) > 2\delta - \hat{t} \geq 2\delta - 1 = \gamma$$

and

$$\hat{\lambda}(y) > 2\delta - \hat{t} \geq 2\delta - 1 = \gamma.$$

Thus x, y $\in$ I and so $x * y \in I$, which implies that

$$\hat{\lambda}(x * y) \geq \delta.$$

Now, if $\hat{t} \wedge \hat{r} \leq \delta$, then

$$\hat{\lambda}(x * y) \geq \delta \geq \hat{t} \wedge \hat{r} > \gamma,$$

which implies that

$$(x * y)_{\hat{t} \wedge \hat{r}} \in_\gamma \hat{\lambda}.$$

If $\hat{t} \wedge \hat{r} > \delta$, then

$$\hat{\lambda}(x * y) + \hat{t} \wedge \hat{r} \geq \delta + \delta = 2\delta,$$

which implies that

$$(x * y)_{\hat{t} \wedge \hat{r}} q_\delta \hat{\lambda}.$$

Hence $\hat{\lambda}$ is an n-dimensional $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X.
Conversely, assume that $\hat{\lambda}$ is an n-dimensional $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. Then I $= \hat{\lambda}_r$. Thus by Theorem 3.2, I is a subalgebra of X. □

Corollary 3.6. *Let $2\delta = 1 + \gamma$ and I be a nonempty subset of a BRK-algebra X. Then I is a subalgebra of X if and only if the characteristic function C_I of I is an n-dimensional $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X.*

Theorem 3.7. *Let $2\delta = 1 + \gamma$ and I be a nonempty subset of a BRK-algebra X. Then I is a subalgebra of X if and only if the n-dimensional fuzzy subset $\hat{\lambda}$ of X defined by*

$$\hat{\lambda}(x) = \begin{cases} \geq \delta & \text{if } x \in I \\ \leq \gamma & \text{if } x \notin I \end{cases}$$

is an n-dimensional $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X.

Proof. Let I be a subalgebra of X. Let x, y $\in$ X and $\hat{t}$, $\hat{r} \in (\gamma, 1]$ be such that

$$x_{\hat{t}} \in_\gamma \vee q_\delta \hat{\lambda} \text{ and } y_{\hat{r}} \in_\gamma \vee q_\delta \hat{\lambda}.$$

which implies that

$$x_{\hat{t}} \in_\gamma \hat{\lambda} \text{ or } x_{\hat{t}} q_\delta \hat{\lambda} \text{ and } y_{\hat{r}} \in_\gamma \hat{\lambda} \text{ or } y_{\hat{r}} q_\delta \hat{\lambda}.$$

If $x_{\hat{t}} \in_\gamma \hat{\lambda}$ and $y_{\hat{r}} q_\delta \hat{\lambda}$, then

$$\hat{\lambda}(x) \geq \hat{t} > \gamma \text{ and } \hat{\lambda}(y) + \hat{r} > 2\delta.$$

This implies that

$$\hat{\lambda}(y) > 2\delta - \hat{r} \geq 2\delta - 1 = \gamma.$$

Which implies that x, y $\in$ I and so x $*$ y $\in$ I. Analogous to Theorem 3.3 and Theorem 3.5 we obtain

$$(x * y)_{\hat{t} \wedge \hat{r}} \in_\gamma \vee q_\delta \hat{\lambda}.$$

Hence $\hat{\lambda}$ is an n-dimensional $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. The other cases can be considered similar to this case.
Conversely, assume that $\hat{\lambda}$ is an n-dimensional $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. Then I $= \hat{\lambda}_r$. Thus by Theorem 3.2, I is a subalgebra of X. □

Corollary 3.8. *Let $2\delta = 1 + \gamma$ and I be a nonempty subset of a BRK-algebra X. Then I is a subalgebra of X if and only if the characteristic function C_I of I is an n-dimensional $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X.*

Theorem 3.9. *Every n-dimensional $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of a BRK-algebra X is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X.*

Proof. Let $\hat{\lambda}$ be an n-dimensional $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. Let x, y $\in$ X and $\hat{t}$, $\hat{r} \in (\gamma, 1]$ be such that

$$x_{\hat{t}} \in_\gamma \hat{\lambda} \text{ and } y_{\hat{r}} \in_\gamma \hat{\lambda}.$$

Then

$$\hat{\lambda}(x) \geq \hat{t} > \gamma \text{ and } \hat{\lambda}(y) \geq \hat{r} > \gamma.$$

Suppose $(x * y)_{\hat{t} \wedge \hat{r}} \overline{\in_\gamma \vee q_\delta} \hat{\lambda}$. Then

$$\hat{\lambda}(x * y) < \hat{t} \wedge \hat{r}$$

and

$$\hat{\lambda}(x * y) + \hat{t} \wedge \hat{r} \leq 2\delta.$$

This implies that

$$\hat{\lambda}(x * y) + \hat{\lambda}(x * y) < \hat{\lambda}(x * y) + \hat{t} \wedge \hat{r} \leq 2\delta.$$

This implies that

$$\hat{\lambda}(x * y) < \delta.$$

Now

$$\hat{\lambda}(x * y) \vee \gamma < \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Choose $\hat{t}_1 \in (\gamma, 1]$ such that

$$2\delta - \{\hat{\lambda}(x * y) \vee \gamma\} \geq \hat{t}_1 > 2\delta - \{\hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta\},$$

that is,

$$(2\delta - \hat{\lambda}(x * y)) \wedge (2\delta - \gamma) \geq \hat{t}_1 > (2\delta - \hat{\lambda}(x)) \vee (2\delta - \hat{\lambda}(y)) \vee \delta.$$

This implies that

$$\hat{t}_1 > 2\delta - \hat{\lambda}(x),\ \hat{t}_1 > 2\delta - \hat{\lambda}(y) \text{ and } 2\delta - \hat{\lambda}(x * y) > \hat{t}_1.$$

This implies that

$$\hat{\lambda}(x) + \hat{t}_1 > 2\delta,\ \hat{\lambda}(y) + \hat{t}_1 > 2\delta \text{ and } \hat{\lambda}(x * y) + \hat{t}_1 < 2\delta.$$

Thus

$$x_{\hat{t}_1} q_\delta \hat{\lambda},\ y_{\hat{t}_1} q_\delta \hat{\lambda} \text{ but } (x * y)_{\hat{t}_1} \overline{\in_\gamma \vee q_\delta} \hat{\lambda},$$

which is a contradiction. Hence $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of X. □

Theorem 3.10. *Every n-dimensional $(\in_\gamma \vee q_\delta, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of a BRK-algebra X is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X.*

Proof. The proof follows from the fact that if $x_{\hat{t}} \in_\gamma \hat{\lambda}$, then $x_{\hat{t}} \in_\gamma \vee q_\delta \hat{\lambda}$. □

Theorem 3.11. *Every n-dimensional fuzzy subalgebra of a BRK-algebra X is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X.*

Proof. Straightforward. □

4 N-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra

In this section, we define the concept of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra in a BRK-algebra and investigate some of their related properties.

Definition 4.1. *An n-dimensional fuzzy set $\hat{\lambda}$ of a BRK-algebra X is called an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X if it satisfies the condition (A), where*
(A) $x_{\hat{t}} \in_\gamma \hat{\lambda}$, $y_{\hat{r}} \in_\gamma \hat{\lambda} \Rightarrow (x * y)_{\hat{t}\wedge\hat{r}} \in_\gamma \vee q_\delta \hat{\lambda}$,
for all $\hat{t}$, $\hat{r} \in (\gamma, 1]$ and for all x, $y \in X$.

Theorem 4.2. *For an n-dimensional fuzzy set $\hat{\lambda}$ of a BRK-algebra X, the condition (A) is equivalent to the condition (B), where*
(B) $\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$,
for all x, $y \in X$.

Proof. (A) $\Rightarrow$ (B)
Suppose (B) does not hold. Then there exist x, y $\in$ X such that

$$\hat{\lambda}(x * y) \vee \gamma < \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Then

$$\hat{\lambda}(x * y) \vee \gamma < \hat{t} \leq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$$

for some $\hat{t} \in (\gamma, \delta]$. Thus

$$x_{\hat{t}} \in_\gamma \hat{\lambda} \text{ and } y_{\hat{t}} \in_\gamma \hat{\lambda}, \text{ but } (x * y)_{\hat{t}} \overline{\in_\gamma \vee q_\delta} \hat{\lambda}.$$

This is a contradiction. Hence

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

(B) $\Rightarrow$ (A)
Let $x_{\hat{t}} \in_\gamma \hat{\lambda}$ and $y_{\hat{r}} \in_\gamma \hat{\lambda}$. Then

$$\hat{\lambda}(x) \geq \hat{t} > \gamma \text{ and } \hat{\lambda}(y) \geq \hat{r} > \gamma.$$

If $(x * y)_{\hat{t}\wedge\hat{r}} \in_\gamma \hat{\lambda}$, then (A) holds.
If $(x * y)_{\hat{t}\wedge\hat{r}} \bar{\in}_\gamma \hat{\lambda}$, then

$$\hat{\lambda}(x * y) < \hat{t} \wedge \hat{r}.$$

Since

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$$

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{t} \wedge \hat{r} \wedge \delta$$

it follows that

$$\hat{\lambda}(x * y) \geq \delta \text{ and } \hat{t} \wedge \hat{r} > \delta.$$

Thus

$$\hat{\lambda}(x * y) + \hat{t} \wedge \hat{r} > \delta + \delta = 2\delta$$

$$\Rightarrow \quad (x * y)_{\hat{t} \wedge \hat{r}} q_\delta \hat{\lambda}.$$

Hence

$$(x * y)_{\hat{t} \wedge \hat{r}} \in_\gamma \vee q_\delta \hat{\lambda}.$$

□

Corollary 4.3. *An n-dimensional fuzzy set $\hat{\lambda}$ of a BRK-algebra X is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X if it satisfies the condition (B).*

Theorem 4.4. *The intersection of any family of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of a BRK-algebra X is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X.*

Proof. Let $\{\hat{\lambda}\}_{i \in I}$ be a family of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebras of a BRK-algebra X and x, y $\in$ X. Then

$$\hat{\lambda}_i(x * y) \vee \gamma \geq \hat{\lambda}_i(x) \wedge \hat{\lambda}_i(y) \wedge \delta$$

for all i $\in$ I. Thus

$$(\wedge_{i \in I} \hat{\lambda}_i)(x * y) \vee \gamma = \wedge_{i \in I} \hat{\lambda}_i(x * y) \vee \gamma$$

$$(\wedge_{i \in I} \hat{\lambda}_i)(x * y) \vee \gamma \geq \wedge_{i \in I} (\hat{\lambda}_i(x) \wedge \hat{\lambda}_i(y) \wedge \delta)$$

$$(\wedge_{i \in I} \hat{\lambda}_i)(x * y) \vee \gamma = (\wedge_{i \in I} \hat{\lambda}_i)(x) \wedge (\wedge_{i \in I} \hat{\lambda}_i)(y) \wedge \delta.$$

Therefore

$$(\wedge_{i \in I} \hat{\lambda}_i)(x * y) \vee \gamma \geq (\wedge_{i \in I} \hat{\lambda}_i)(x) \wedge (\wedge_{i \in I} \hat{\lambda}_i)(y) \wedge \delta.$$

Hence, $\wedge_{i \in I} \hat{\lambda}_i$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. □

Definition 4.5. *For n-dimensional fuzzy set $\hat{\lambda}$ of BRK-algebra X. We defined the following:*

$$\hat{\lambda}^\gamma_{\hat{r}} = \{x \in X | x_{\hat{r}} \in_\gamma \hat{\lambda}\}$$

$$\hat{\lambda}^\delta_{\hat{r}} = \{x \in X | x_{\hat{r}} q_\delta \hat{\lambda}\}$$

and

$$[\hat{\lambda}]^\delta_{\hat{r}} = \{x \in X | x_{\hat{r}} \in_\gamma \vee q_\delta \hat{\lambda}\} \text{ for all } \hat{r} \in [0,1].$$

It is clear that

$$[\hat{\lambda}]^\delta_{\hat{r}} = \hat{\lambda}^\gamma_{\hat{r}} \cup \hat{\lambda}^\delta_{\hat{r}}.$$

The relationship between n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebras and the crisp subalgebras in a BRK-algebra X can be expressed in the form of the following theorem.

Theorem 4.6. *Let $\hat{\lambda}$ be an n-dimensional fuzzy set of a BRK-algebra X. Then $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X if and only if $\hat{\lambda}^\gamma_{\hat{r}}(\neq \phi)$ is a subalgebra of X for all $\hat{r} \in (\gamma, \delta]$.*

Proof. Let $\hat{\lambda}$ be an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X and x, y $\in \hat{\lambda}^\gamma_{\hat{r}}$ for $\hat{r} \in (\gamma, \delta]$. Then

$$\hat{\lambda}(x) \geq \hat{r} > \gamma \text{ and } \hat{\lambda}(y) \geq \hat{r} > \gamma.$$

Since

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$$

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{r} \wedge \hat{r} \wedge \delta$$

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{r} \wedge \delta$$

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{r} > \gamma$$

we have

$$\hat{\lambda}(x * y) \geq \hat{r}.$$

Thus

$$x * y \in \hat{\lambda}^\gamma_{\hat{r}}.$$

Therefore $\hat{\lambda}^\gamma_{\hat{r}}$ is a subalgebra of X.
Conversely, assume that $\hat{\lambda}^\gamma_{\hat{r}}$ is a subalgebra of X for all $\hat{r} \in (\gamma, \delta]$. Suppose x, y $\in$ X are such that

$$\hat{\lambda}(x * y) \vee \gamma < \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Select $\hat{t} \in (\gamma, \delta]$ such that

$$\hat{\lambda}(x * y) \vee \gamma < \hat{t} = \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Then

$$x_{\hat{t}} \in_\gamma \hat{\lambda},\ y_{\hat{t}} \in_\gamma \hat{\lambda}, \text{ but } (x * y)_{\hat{t}} \overline{\in_\gamma \vee q_\delta} \hat{\lambda}.$$

which is a contradiction. Hence

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Therefore $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. $\square$

By setting $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.6, the following corollary is obtained.

Corollary 4.7. *Let $\hat{\lambda}$ be an n-dimensional fuzzy set of a BRK-algebra X. Then $\hat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy subalgebra of X if and only if $\hat{\lambda}_{\hat{r}}(\neq \phi)$ is a subalgebra of X for all $\hat{r} \in (0, 0.5]$.*

Theorem 4.8. *Let $\hat{\lambda}$ be an n-dimensional fuzzy set of a BRK-algebra X. If $2\delta = 1 + \gamma$, then $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X if and only if $\hat{\lambda}^{\delta}_{\hat{r}}(\neq \phi)$ is a subalgebra of X for all $\hat{r} \in (\delta, 1]$.*

Proof. Suppose $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. Let $x, y \in \hat{\lambda}^{\delta}_{\hat{r}}$. Then

$$x_{\hat{r}} q_\delta \hat{\lambda},\ y_{\hat{r}} q_\delta \hat{\lambda}.$$

This implies that

$$\hat{\lambda}(x) + \hat{r} > 2\delta,\ \hat{\lambda}(y) + \hat{r} > 2\delta$$

$$\hat{\lambda}(x) > 2\delta - \hat{r},\ \hat{\lambda}(y) > 2\delta - \hat{r}$$

$$\hat{\lambda}(x) > 2\delta - \hat{r} \geq 2\delta - 1 = \gamma,\ \hat{\lambda}(y) > 2\delta - \hat{r} \geq 2\delta - 1 = \gamma.$$

By hypothesis

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$$

$$\hat{\lambda}(x * y) \vee \gamma > (2\delta - \hat{r}) \wedge (2\delta - \hat{r}) \wedge \delta$$

$$\hat{\lambda}(x * y) \vee \gamma = (2\delta - \hat{r}) \wedge \delta$$

$$\hat{\lambda}(x * y) \vee \gamma > (2\delta - \hat{r}) \qquad (\because \hat{r} > \delta)$$

So

$$\hat{\lambda}(x * y) > 2\delta - \hat{r}.$$

Thus

$$\hat{\lambda}(x * y) + \hat{r} > 2\delta.$$

This implies that

$$(x * y)_{\hat{r}} q_\delta \hat{\lambda},$$

that is

$$x * y \in \hat{\lambda}^\delta_{\hat{r}}.$$

Hence $\hat{\lambda}^\delta_{\hat{r}}$ is a subalgebra of X.
Conversely, assume that $\hat{\lambda}^\delta_{\hat{r}}$ is a subalgebra of X for all $\hat{r} \in (\delta, 1]$. Now, suppose x, y $\in$ X are such that

$$\hat{\lambda}(x * y) \vee \gamma < \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Then

$$2\delta - (\hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta) < 2\delta - (\hat{\lambda}(x * y) \vee \gamma).$$

This implies that

$$(2\delta - \hat{\lambda}(x)) \vee (2\delta - \hat{\lambda}(y)) \vee \delta < (2\delta - \hat{\lambda}(x * y)) \wedge (2\delta - \gamma).$$

Select some $\hat{r} \in (\delta, 1]$ such that

$$(2\delta - \hat{\lambda}(x)) \vee (2\delta - \hat{\lambda}(y)) \vee \delta < \hat{r} \leq (2\delta - \hat{\lambda}(x * y)) \wedge (2\delta - \gamma).$$

Then

$$2\delta - \hat{\lambda}(x) < \hat{r},\ 2\delta - \hat{\lambda}(y) < \hat{r} \text{ and } \hat{r} \leq 2\delta - \hat{\lambda}(x * y)$$

$$\Rightarrow \qquad \hat{\lambda}(x) + \hat{r} > 2\delta,\ \hat{\lambda}(y) + \hat{r} > 2\delta \text{ and } \hat{\lambda}(x * y) + \hat{r} \leq 2\delta.$$

Thus

$$x_{\hat{r}} q_\delta \hat{\lambda},\ y_{\hat{r}} q_\delta \hat{\lambda}, \text{ but } (x * y)_{\hat{r}} \bar{q_\delta} \hat{\lambda},$$

that is, x and y are in $\hat{\lambda}^\delta_{\hat{r}}$ but $x * y \notin \hat{\lambda}^\delta_{\hat{r}}$, a contradiction. Hence

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

This shows that $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. □

By setting $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.8, the following corollary is obtained.

Corollary 4.9. *Let $\hat{\lambda}$ be an n-dimensional fuzzy set of a BRK-algebra X. Then $\hat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy subalgebra of X if and only if $Q(\hat{\lambda}; \hat{r})(\neq \phi)$ is a subalgebra of X for all $\hat{r} \in (0.5, 1]$, where*

$$Q(\hat{\lambda}; \hat{r}) = \{x \in X | x_{\hat{r}} q \hat{\lambda}\}.$$

Theorem 4.10. *Let $\hat{\lambda}$ be an n-dimensional fuzzy set of a BRK-algebra X. If $2\delta = 1 + \gamma$, then $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X if and only if $[\hat{\lambda}]^\delta_{\hat{r}} (\neq \phi)$ is a subalgebra of X for all $\hat{r} \in (\gamma, 1]$.*

Proof. Let $\hat{\lambda}$ be an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X and $\hat{r} \in (\gamma, 1]$. Let $x, y \in [\hat{\lambda}]^\delta_{\hat{r}}$, so we have

$$x_{\hat{r}}, y_{\hat{r}} \in_\gamma \vee q_\delta \hat{\lambda},$$

that is

$$\hat{\lambda}(x) \geq \hat{r} > \gamma \text{ or } \hat{\lambda}(x) > 2\delta - \hat{r} > 2\delta - 1 = \gamma \tag{1}$$

and

$$\hat{\lambda}(y) \geq \hat{r} > \gamma \text{ or } \hat{\lambda}(y) > 2\delta - \hat{r} > 2\delta - 1 = \gamma \tag{2}$$

Case 1: If $\hat{r} \in (\gamma, \delta]$, then

$$2\delta - \hat{r} \geq \delta \geq \hat{r}.$$

Thus it follows from (1) and (2) that

$$\hat{\lambda}(x) \geq \hat{r} > \gamma \text{ and } \hat{\lambda}(y) \geq \hat{r} > \gamma.$$

By hypothesis

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

and this implies

$$\hat{\lambda}(x * y) \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$$

$$\hat{\lambda}(x * y) \geq \hat{r} \wedge \hat{r} \wedge \delta$$

$$\hat{\lambda}(x * y) = \hat{r} \wedge \delta$$

$$\hat{\lambda}(x * y) = \hat{r} > \gamma.$$

Hence

$$(x * y)_{\hat{r}} \in_\gamma \hat{\lambda}.$$

Case 2: If $\hat{r} \in (\gamma, 1]$, then

$$2\delta - \hat{r} < \delta < \hat{r}.$$

Thus it follows from (1) and (2) that

$$\hat{\lambda}(x) > 2\delta - \hat{r} \text{ and } \hat{\lambda}(y) > 2\delta - \hat{r}.$$

By hypothesis

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$$

and this implies

$$\hat{\lambda}(x * y) \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta$$

$$\hat{\lambda}(x * y) \geq (2\delta - \hat{r}) \wedge (2\delta - \hat{r}) \wedge \delta$$

$$\hat{\lambda}(x * y) \geq (2\delta - \hat{r}) \wedge \delta$$

$$\hat{\lambda}(x * y) = (2\delta - \hat{r}).$$

Thus

$$(x * y)_{\hat{r}} q_\delta \hat{\lambda}.$$

Hence

$$(x * y)_{\hat{r}} \in_\gamma \vee q_\delta \hat{\lambda},$$

that is,

$$x * y \in [\hat{\lambda}]^\delta_{\hat{r}}.$$

This shows that $[\hat{\lambda}]^\delta_{\hat{r}}$ is a subalgebra of X.
Conversely, suppose that $[\hat{\lambda}]^\delta_{\hat{r}}$ is a subalgebra of X for all $\hat{r} \in (\gamma, 1]$. Suppose x, y $\in$ X are such that

$$\hat{\lambda}(x * y) \vee \gamma < \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Select some $\hat{r} \in (\gamma, 1]$ such that

$$\hat{\lambda}(x * y) \vee \gamma < \hat{r} = \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Then

$$x_{\hat{r}} \in_\gamma \hat{\lambda},\ y_{\hat{r}} \in_\gamma \hat{\lambda} \text{ but } (x * y)_{\hat{r}} \overline{\in_\gamma \vee q_\delta} \hat{\lambda}.$$

This is a contradiction. Hence

$$\hat{\lambda}(x * y) \vee \gamma \geq \hat{\lambda}(x) \wedge \hat{\lambda}(y) \wedge \delta.$$

Therefore $\hat{\lambda}$ is an n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra of X. □

By setting $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.10, the following corollary is obtained.

Corollary 4.11. *Let $\hat{\lambda}$ be an n-dimensional fuzzy set of a BRK-algebra X. Then $\hat{\lambda}$ is an n-dimensional $(\in, \in \vee q)$-fuzzy subalgebra of X if and only if $[\hat{\lambda}]_{\hat{r}} (\neq \phi)$ is a subalgebra of X for all $\hat{r} \in (0, 1]$.*

5 Conclusion

In this paper, we apply the fuzzy set and idea of belongingness and quasi-coincidence to the n-dimensional fuzzy subalgebras in BRK-algebras. And in fact, some results in this paper have already constituted a platform for further discussion concerning other algebraic structures (such as BL-algebras, R0-algebras etc.). To consider these results to some possible applications in computer sciences and information systems in the future. The purpose of the present paper is to introduce the concept of n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra in BRK-algebra and investigate some of their related properties. We also prove that the relationship between n-dimensional $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy subalgebra and the crisp subalgebra in BRK-algebra are discussed.

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