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Integral theorems for the quaternionic G-monogenic mappings

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Abstract

In the paper [1] considered a new class of quaternionic mappings, so-called G-monogenic mappings. In this paper we prove analogues of classical integral theorems of the holomorphic function theory: the Cauchy integral theorems for surface and curvilinear integrals, and the Cauchy integral formula for G-monogenic mappings.

Introduction. The Cauchy integral theorem and Cauchy integral formula for holomorphic functions of the complex variable are fundamental results of the classical complex analysis. Analogues of these results are also important tools in the quaternionic analysis.

Maybe the first quaternionic analogues of the mentioned results for a surface in $\mathbb{R}^3$ are obtained by G. Moisil and N. Theodoresco [2]. Namely, they proved some analogues of the Cauchy's theorems for a smooth surface and for continuously differentiable functions f satisfying the equality

$$\frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k = 0,$$

where i, j, k are the basis quaternionic units.

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R. Fueter [3] transferred the results of paper [2] for a smooth surface in $\mathbb{R}^4$ and for so-called the *regular* functions f which by definition satisfies the equality

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = 0.$$

The proofs of [2] and [3] are based on the Stokes formula.

B. Schuler [4] generalized the results of Fueter by adapting of Goursat's proof of the Cauchy's theorem. Due to this, he replaced the condition of continuity of partial derivatives to the differentiability of the real-valued components in the Stolz sense.

A. Sudbery [5] proved the Cauchy theorem and Cauchy formula under more general conditions on a function and a surface. He assumed as in the complex analysis only differentiability of real-valued components of the integrand. He also considered the so-called differentiable surface and the rectifiable 3-chain.

O. Herus [6] proved the quaternionic Cauchy theorem for a surface in $\mathbb{R}^3$ under the same conditions on the function as in [5], but for another class of surfaces.

Some analogues of the quaternionic Cauchy theorem and Cauchy integral formula in the theory of *s-regular* functions are established in [7].

In this paper for quaternionic *G-monogenic* mappings we prove analogues of the Cauchy theorem for surface and curvilinear integrals and the Cauchy integral formula curvilinear integral.

1. The algebra of complex quaternion $\mathbb{H}(\mathbb{C})$ and G-monogenic mappings.

Let $\mathbb{H}(\mathbb{C})$ be the quaternion algebra over the field of complex numbers $\mathbb{C}$, whose basis consists of the unit 1 of the algebra and of the elements I, J, K satisfying the multiplication rules:

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J.$$

In the algebra $\mathbb{H}(\mathbb{C})$ there exists another basis $\{e_1, e_2, e_3, e_4\}$:

$$e_1 = \frac{1}{2}(1 + iI), \quad e_2 = \frac{1}{2}(1 - iI), \quad e_3 = \frac{1}{2}(iJ - K), \quad e_4 = \frac{1}{2}(iJ + K),$$

where i is the complex imaginary unit. Multiplication table in a new basis can be represented as

$\cdot$	e_1	e_2	e_3	e_4
e_1	e_1	0	e_3	0
e_2	0	e_2	0	e_4
e_3	0	e_3	0	e_1
e_4	e_4	0	e_2	0

. (1)

The unit 1 can be decomposed as $1 = e_1 + e_2$.

Consider linear functionals $f_1 : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ and $f_2 : \mathbb{H}(\mathbb{C}) \to \mathbb{C}$ satisfying the equalities

$$f_1(e_1) = f_1(e_3) = 1, \qquad f_1(e_2) = f_1(e_4) = 0,$$

$$f_2(e_2) = f_2(e_4) = 1, \qquad f_2(e_1) = f_2(e_3) = 0.$$

Let us consider the vectors

$$i_1 = 1 = e_1 + e_2, \quad i_2 = a_1 e_1 + a_2 e_2, \quad i_3 = b_1 e_1 + b_2 e_2,$$

$a_k, b_k \in \mathbb{C}$, $k = 1, 2$, which are a linearly independent over the field of real numbers $\mathbb{R}$ (see [8, p. 223]). It means that the equality

$$\alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_3 = 0, \qquad \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$$

holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

In the algebra $\mathbb{H}(\mathbb{C})$ consider the linear span $E_3 := \{\zeta = xi_1 + yi_2 + zi_3 : x, y, z \in \mathbb{R}\}$ generated by the vectors i_1, i_2, i_3 over the field $\mathbb{R}$. Denote $f_k(E_3) := \{f_k(\zeta) : \zeta \in E_3\}$, $k = 1, 2$. In what follows, we make the following essential assumption: $f_1(E_3) = f_2(E_3) = \mathbb{C}$. Obviously, it holds if and only if at least one of the numbers in each of pairs (a_1, b_1) or (a_2, b_2) belongs to $\mathbb{C} \setminus \mathbb{R}$.

Let us introduce the notations

$$\xi_1 := f_1(\zeta) = x + ya_1 + zb_1, \qquad \xi_2 := f_2(\zeta) = x + ya_2 + zb_2.$$

Now, the element $\zeta \in E_3$ can be represented in the form $\zeta = \xi_1 e_1 + \xi_2 e_2$.

A set $S \subset \mathbb{R}^3$ is associated with the set $S_\zeta := \{\zeta = xi_1 + yi_2 + zi_3 : (x, y, z) \in S\}$ in E_3. We also note that a topological property of a set S_ζ in E_3 understand as the same topological property of the set S in $\mathbb{R}^3$. For example, we will say that a curve $\gamma_\zeta \subset E_3$ is homotopic to the zero if $\gamma \subset \mathbb{R}^3$ is homotopic to the zero, etc. Let Ω_ζ be a domain in E_3.

A continuous mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ (or $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$) is *right-G-monogenic* (or resp. *left-G-monogenic*) in a domain $\Omega_\zeta \subset E_3$, if Φ (or resp. $\widehat{\Phi}$) is differentiable in the sense of the Gateaux in every point of Ω_ζ, i. e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{H}(\mathbb{C})$ (or resp. $\widehat{\Phi}'(\zeta) \in \mathbb{H}(\mathbb{C})$) such that

$$\lim_{\varepsilon \to 0+0} \Big(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\Big)\varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_3$$

$$\left(\text{or resp. } \lim_{\varepsilon \to 0+0} \left(\widehat{\Phi}(\zeta + \varepsilon h) - \widehat{\Phi}(\zeta)\right)\varepsilon^{-1} = \widehat{\Phi}'(\zeta)h \quad \forall\, h \in E_3\right).$$

$\Phi'(\zeta)$ is *the right Gateaux derivative* in the point ζ and $\widehat{\Phi}'(\zeta)$ is *the left Gateaux derivative* in the point ζ.

A mapping $\Phi(\zeta)$ (or $\widehat{\Phi}(\zeta)$) of the variable $\zeta = x + yi_2 + zi_3 \in \Omega_\zeta$ with differentiable real–valued components is right-G-monogenic (or resp. left-G-monogenic) if and only if the following Cauchy–Riemann conditions are satisfied [1]:

$$\frac{\partial \Phi}{\partial y} = i_2 \frac{\partial \Phi}{\partial x}, \qquad \frac{\partial \Phi}{\partial z} = i_3 \frac{\partial \Phi}{\partial x} \tag{2}$$

or resp.

$$\frac{\partial \widehat{\Phi}}{\partial y} = \frac{\partial \widehat{\Phi}}{\partial x} i_2, \qquad \frac{\partial \widehat{\Phi}}{\partial z} = \frac{\partial \widehat{\Phi}}{\partial x} i_3. \tag{3}$$

It follows from the decomposition of the resolvent

$$(t-\zeta)^{-1} = \frac{1}{t-\xi_1}\, e_1 + \frac{1}{t-\xi_2}\, e_2, \qquad \forall\, t \in \mathbb{C}:\ t \neq \xi_1,\ t \neq \xi_2$$

that the points $(x, y, z) \in \mathbb{R}^3$ corresponding to the noninvertible elements $\zeta = xi_1 + yi_2 + zi_3$ of the algebra $\mathbb{H}(\mathbb{C})$ form the straight lines in $\mathbb{R}^3$:

$$L^1:\ x + y\mathrm{Re}\, a_1 + z\mathrm{Re}\, b_1 = 0, \qquad y\mathrm{Im}\, a_1 + z\mathrm{Im}\, b_1 = 0,$$

$$L^2:\ x + y\mathrm{Re}\, a_2 + z\mathrm{Re}\, b_2 = 0, \qquad y\mathrm{Im}\, a_2 + z\mathrm{Im}\, b_2 = 0$$

in the three-dimensional space $\mathbb{R}^3$.

Denote by $D_k \subset \mathbb{C}$ the image of Ω_ζ under the mapping $f_k,\ k = 1, 2$. A constructive description of all right- and left-G-monogenic mappings by means of holomorphic functions of the complex variable are obtained in the paper [1]. Namely, proved the theorem:

Let a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight lines L^1, L^2 and $f_1(E_3) = f_2(E_3) = \mathbb{C}$. Then any right-$G$-monogenic mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ can be expressed in the form

$$\Phi(\zeta) = F_1(\xi_1)e_1 + F_2(\xi_2)e_2 + F_3(\xi_1)e_3 + F_4(\xi_2)e_4 \tag{4}$$

$$\forall\, \zeta = xi_1 + yi_2 + zi_3 \in \Omega_\zeta,$$

where F_1, F_4 are the certain holomorphic in a domain D_1 functions of the variable $\xi_1 := x + ya_1 + zb_1$ and F_2, F_3 are the certain holomorphic in a domain D_2 functions of the variable $\xi_2 := x + ya_2 + zb_2$.

Under the same assumptions, any left-G-monogenic mapping $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ can be expressed in the form

$$\widehat{\Phi}(\zeta) = \widehat{F}_1(\xi_1)e_1 + \widehat{F}_2(\xi_2)e_2 + \widehat{F}_3(\xi_2)e_3 + \widehat{F}_4(\xi_1)e_4, \tag{5}$$

where $\widehat{F}_n$, $n=\overline{1,4}$ are defined similarly to above.

2. Cauchy integral theorem for a curvilinear integral.

Let γ be a Jordan rectifiable curve in $\mathbb{R}^3$. For a continuous mapping $\Psi:\gamma_\zeta\to\mathbb{H}(\mathbb{C})$ of the form

$$\Psi(\zeta)=\sum_{k=1}^{4}U_k(x,y,z)e_k+i\sum_{k=1}^{4}V_k(x,y,z)e_k, \tag{6}$$

where $(x,y,z)\in\gamma$ and $U_k:\gamma\to\mathbb{R}$, $V_k:\gamma\to\mathbb{R}$, we define integrals along a Jordan rectifiable curve γ_ζ by the equalities:

$$\int\limits_{\gamma_\zeta}d\zeta\Psi(\zeta):=\sum_{k=1}^{4}e_k\int\limits_{\gamma}U_k(x,y,z)dx+\sum_{k=1}^{4}i_2e_k\int\limits_{\gamma}U_k(x,y,z)dy+$$

$$+\sum_{k=1}^{4}i_3e_k\int\limits_{\gamma}U_k(x,y,z)dz+i\sum_{k=1}^{4}e_k\int\limits_{\gamma}V_k(x,y,z)dx+$$

$$+i\sum_{k=1}^{4}i_2e_k\int\limits_{\gamma}V_k(x,y,z)dy+i\sum_{k=1}^{4}i_3e_k\int\limits_{\gamma}V_k(x,y,z)dz$$

and

$$\int\limits_{\gamma_\zeta}\Psi(\zeta)d\zeta:=\sum_{k=1}^{4}e_k\int\limits_{\gamma}U_k(x,y,z)dx+\sum_{k=1}^{4}e_ki_2\int\limits_{\gamma}U_k(x,y,z)dy+$$

$$+\sum_{k=1}^{4}e_ki_3\int\limits_{\gamma}U_k(x,y,z)dz+i\sum_{k=1}^{4}e_k\int\limits_{\gamma}V_k(x,y,z)dx+$$

$$+i\sum_{k=1}^{4}e_ki_2\int\limits_{\gamma}V_k(x,y,z)dy+i\sum_{k=1}^{4}e_ki_3\int\limits_{\gamma}V_k(x,y,z)dz,$$

where $d\zeta:=dx+i_2dy+i_3dz$.

Let Σ be a piece-smooth surface in $\mathbb{R}^3$. For a continuous function $\Psi:\Sigma_\zeta\to\mathbb{H}(\mathbb{C})$ of the form (6), where $(x,y,z)\in\Sigma$, we define surface integrals on Σ_ζ with the differential form $\sigma:=dydz+dzdxi_2+dxdyi_3$ by the equalities

$$\int\limits_{\Sigma_\zeta}\sigma\Psi(\zeta):=\sum_{k=1}^{4}e_k\int\limits_{\Sigma}U_k(x,y,z)dydz+\sum_{k=1}^{4}i_2e_k\int\limits_{\Sigma}U_k(x,y,z)dzdx+$$

$$+\sum_{k=1}^{4} i_3 e_k \int\limits_{\Sigma} U_k(x,y,z)dxdy + i\sum_{k=1}^{4} e_k \int\limits_{\Sigma} V_k(x,y,z)dydz+$$

$$+i\sum_{k=1}^{4} i_2 e_k \int\limits_{\Sigma} V_k(x,y,z)dzdx + i\sum_{k=1}^{4} i_3 e_k \int\limits_{\Sigma} V_k(x,y,z)dxdy;$$

$$\int\limits_{\Sigma_\zeta} \Psi(\zeta)\sigma := \sum_{k=1}^{4} e_k \int\limits_{\Sigma} U_k(x,y,z)dydz + \sum_{k=1}^{4} e_k i_2 \int\limits_{\Sigma} U_k(x,y,z)dzdx+$$

$$+\sum_{k=1}^{4} e_k i_3 \int\limits_{\Sigma} U_k(x,y,z)dxdy + i\sum_{k=1}^{4} e_k \int\limits_{\Sigma} V_k(x,y,z)dydz+$$

$$+i\sum_{k=1}^{4} e_k i_2 \int\limits_{\Sigma} V_k(x,y,z)dzdx + i\sum_{k=1}^{4} e_k i_3 \int\limits_{\Sigma} V_k(x,y,z)dxdy.$$

If a mapping $\Psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is continuous together with partial derivatives of the first order in a domain Ω_ζ, Σ is a piece-smooth surface in Ω and the edge γ of surface Σ is a rectifiable Jordan curve, then the following analogues of the Stokes formula are true:

$$\int\limits_{\gamma_\zeta} d\zeta\Psi(\zeta) = \int\limits_{\Sigma_\zeta} \left(i_2\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi}{\partial y}\right) dxdy + \left(i_3\frac{\partial\Psi}{\partial y} - i_2\frac{\partial\Psi}{\partial z}\right) dydz+$$

$$+\left(\frac{\partial\Psi}{\partial z} - i_3\frac{\partial\Psi}{\partial x}\right) dzdx, \tag{7}$$

$$\int\limits_{\gamma_\zeta} \Psi(\zeta)d\zeta = \int\limits_{\Sigma_\zeta} \left(\frac{\partial\Psi}{\partial x}i_2 - \frac{\partial\Psi}{\partial y}\right) dxdy + \left(\frac{\partial\Psi}{\partial y}i_3 - \frac{\partial\Psi}{\partial z}i_2\right) dydz+$$

$$+\left(\frac{\partial\Psi}{\partial z} - \frac{\partial\Psi}{\partial x}i_3\right) dzdx. \tag{8}$$

Now, the next theorem is a result of the formulas (7), (8) and the equalities (2), (3), respectively.

Theorem 1. *Suppose that $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a right-G-monogenic mapping in a domain Ω_ζ and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a left-G-monogenic mapping in*

Ω_ζ. *Suppose also that* Σ *is a piece-smooth surface in* Ω, *and the edge* γ *of surface* Σ *is a rectifiable Jordan curve. Then*

$$\int\limits_{\gamma_\zeta} d\zeta\, \Phi(\zeta) = \int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta) d\zeta = 0. \tag{9}$$

In the case where a domain Ω is convex, then by the usual way (see, e. g., [9]) the equality (9) can be prove for an arbitrary closed Jordan rectifiable curve γ_ζ.

In the case where a domain Ω is an arbitrary, then similarly to the proof of Theorem 3.2 [10] we can prove the following

Theorem 2. *Let* $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *be a right-G-monogenic mapping in a domain* Ω_ζ *and* $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *be a left-G-monogenic mapping in* Ω_ζ . *Then for every closed Jordan rectifiable curve* γ_ζ *homotopic to a point in* Ω_ζ , *the the equalities* (9) *holds.*

3. Cauchy integral formula.

To establish the Cauchy integral formula for a curvilinear integral, consider the following auxiliary statement.

Lemma 1. *Suppose that a domain* $\Omega \subset \mathbb{R}^3$ *is convex in the direction of the straight lines* L^1, L^2 *and* $f_1(E_3) = f_2(E_3) = \mathbb{C}$. *Suppose also that* $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is a right-G-monogenic mapping in* Ω_ζ, *and* $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ *is a left-G-monogenic mapping in* Ω_ζ, *and* γ_ζ *is an arbitrary rectifiable curve in* Ω_ζ. *Then*

$$\int\limits_{\gamma_\zeta} d\zeta\, \Phi(\zeta) = e_1 \int\limits_{\gamma_1} F_1(\xi_1) d\xi_1 + e_2 \int\limits_{\gamma_2} F_2(\xi_2) d\xi_2 + e_3 \int\limits_{\gamma_1} F_3(\xi_1) d\xi_1 + e_4 \int\limits_{\gamma_2} F_4(\xi_2) d\xi_2\,, \tag{10}$$

and respectively

$$\int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta) d\zeta = e_1 \int\limits_{\gamma_1} \widehat{F}_1(\xi_1) d\xi_1 + e_2 \int\limits_{\gamma_2} \widehat{F}_2(\xi_2) d\xi_2 + e_3 \int\limits_{\gamma_2} \widehat{F}_3(\xi_2) d\xi_2 + e_4 \int\limits_{\gamma_1} \widehat{F}_4(\xi_1) d\xi_1\,, \tag{11}$$

where γ_1, γ_2 *are the images of* γ_ζ *under the mappings* f_1, f_2 *and* F_n, $\widehat{F}_n$, $n = \overline{1,4}$ *are the same functions as in* (4) *and* (5) *respectively.*

Proof. The equality (10) follows immediately from the representation (4), the equality $d\zeta = d\xi_1 e_1 + d\xi_2 e_2$ and the multiplication rules (1). Similarly we can prove the equality (11). The Lemma is proved.

Let $\zeta \in E_3$. An inverse element ζ^{-1} is of the following form:

$$\zeta^{-1} = \frac{1}{\xi_1} e_1 + \frac{1}{\xi_2} e_2 \tag{12}$$

and it exists if and only if $\xi_1 \neq 0$ and $\xi_2 \neq 0$.

Let $\zeta_0 = \xi_1^{(0)} e_1 + \xi_2^{(0)} e_2$ be a point in a domain $\Omega_\zeta \subset E_3$. In a neighborhood of ζ_0 contained in Ω_ζ let us take a circle $C(\zeta_0)$ with the center at the point ζ_0. By $C_k \subset \mathbb{C}$ we denote the image of $C(\zeta_0)$ under the mapping $f_k,\ k = 1, 2$. We assume that the circle $C(\zeta_0)$ *embraces the set* $\{\zeta - \zeta_0 : \zeta \in L_\zeta^1 \cup L_\zeta^2\}$. It means that C_k bounds some domain D_k' and $\xi_k^{(0)} \in D_k',\ \ k = 1, 2$.

We say that the curve $\gamma_\zeta \subset \Omega_\zeta$ *embraces once the set* $\{\zeta - \zeta_0 : \zeta \in L_\zeta^1 \cup L_\zeta^2\}$, if there exists a circle $C(\zeta_0)$ which embraces the mentioned set and is homotopic to γ_ζ in the domain $\Omega_\zeta \setminus \{\zeta - \zeta_0 : \zeta \in L_\zeta^1 \cup L_\zeta^2\}$.

Theorem 3. *Suppose that a domain $\Omega \subset \mathbb{R}^3$ is convex in the direction of the straight lines L^1, L^2 and $f_1(E_3) = f_2(E_3) = \mathbb{C}$. Suppose also that $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a right-G-monogenic mapping in Ω_ζ and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a left-G-monogenic mapping in Ω_ζ. Then for every point $\zeta_0 \in \Omega_\zeta$ the following equalities are true:*

$$\Phi(\zeta_0) = \frac{1}{2\pi i} \int\limits_{\gamma_\zeta} (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta) \tag{13}$$

and

$$\widehat{\Phi}(\zeta_0) = \frac{1}{2\pi i} \int\limits_{\gamma_\zeta} \widehat{\Phi}(\zeta)(\zeta - \zeta_0)^{-1} d\zeta, \tag{14}$$

where γ_ζ is an arbitrary closed Jordan rectifiable curve in Ω_ζ, that embraces once the set $\{\zeta - \zeta_0 : \zeta \in L_\zeta^1 \cup L_\zeta^2\}$.

Proof. Inasmuch as γ_ζ is homotopic to $C(\zeta_0)$ in the domain $\Omega_\zeta \setminus \{\zeta - \zeta_0 : \zeta \in L_\zeta^1 \cup L_\zeta^2\}$, it follows from Theorem 2 that

$$\frac{1}{2\pi i} \int\limits_{\gamma_\zeta} (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{C(\zeta_0)} (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta).$$

Further, using the equality (12), Lemma 1 and the integral Cauchy formula for holomorphic functions F_n, $n = \overline{1, 4}$, we obtain immediately the following equalities:

$$\frac{1}{2\pi i} \int\limits_{C(\zeta_0)} (\zeta - \zeta_0)^{-1} d\zeta\, \Phi(\zeta) = e_1 \frac{1}{2\pi i} \int\limits_{C_1} \frac{F_1(\xi_1)}{\xi_1 - \xi_1^{(0)}} d\xi_1 + e_2 \frac{1}{2\pi i} \int\limits_{C_2} \frac{F_2(\xi_2)}{\xi_2 - \xi_2^{(0)}} d\xi_2 +$$

$$+ e_3 \frac{1}{2\pi i} \int\limits_{C_1} \frac{F_3(\xi_1)}{\xi_1 - \xi_1^{(0)}} d\xi_1 + e_4 \frac{1}{2\pi i} \int\limits_{C_2} \frac{F_4(\xi_2)}{\xi_2 - \xi_2^{(0)}} d\xi_2 =$$

$$= F_1\big(\xi_1^{(0)}\big)e_1 + F_2\big(\xi_2^{(0)}\big)e_2 + F_3\big(\xi_1^{(0)}\big)e_3 + F_4\big(\xi_2^{(0)}\big)e_4 = \Phi(\zeta_0),$$

where $\zeta_0 = \xi_1^{(0)}e_1 + \xi_2^{(0)}e_2$. Similarly can be proved the equality (14). The Theorem is proved.

We note that the method of this proof is similarly to the proof of the Theorem 6 of the paper [8], where Cauchy integral formula is obtained in a finite-dimensional semi-simple commutative algebra.

4. Cauchy integral theorem for a surface integral.

Let Ω be a bounded domain in $\mathbb{R}^3$. For a continuous mapping $\Psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ of the form (6), where $(x, y, z) \in \Omega$ and $U_k : \Omega \to \mathbb{R}$, $V_k : \Omega \to \mathbb{R}$, we define a volume integral by the equality

$$\int\limits_{\Omega_\zeta} \Psi(\zeta)dxdydz := \sum_{k=1}^{4} e_k \int\limits_{\Omega} U_k(x, y, z)dxdydz + i\sum_{k=1}^{4} e_k \int\limits_{\Omega} V_k(x, y, z)dxdydz.$$

Let Σ be a piece-smooth surface in $\mathbb{R}^3$. For a continuous mapping $\Psi : \Sigma_\zeta \to \mathbb{H}(\mathbb{C})$ of the form (6), where $(x, y, z) \in \Sigma$ and $U_k : \Sigma \to \mathbb{R}$, $V_k : \Sigma \to \mathbb{R}$, we define the surface integrals on a piece-smooth surface Σ_ζ with the differential form $\sigma := dydz + dzdxi_2 + dxdyi_3$ by the equalities

$$\int\limits_{\Sigma_\zeta} \sigma\Psi(\zeta) := \sum_{k=1}^{4} e_k \int\limits_{\Sigma} U_k(x, y, z)dydz + \sum_{k=1}^{4} i_2e_k \int\limits_{\Sigma} U_k(x, y, z)dzdx+$$

$$+\sum_{k=1}^{4} i_3e_k \int\limits_{\Sigma} U_k(x, y, z)dxdy + i\sum_{k=1}^{4} e_k \int\limits_{\Sigma} V_k(x, y, z)dydz+$$

$$+i\sum_{k=1}^{4} i_2e_k \int\limits_{\Sigma} V_k(x, y, z)dzdx + i\sum_{k=1}^{4} i_3e_k \int\limits_{\Sigma} V_k(x, y, z)dxdy$$

and

$$\int\limits_{\Sigma_\zeta} \Psi(\zeta)\sigma := \sum_{k=1}^{4} e_k \int\limits_{\Sigma} U_k(x, y, z)dydz + \sum_{k=1}^{4} e_ki_2 \int\limits_{\Sigma} U_k(x, y, z)dzdx+$$

$$+\sum_{k=1}^{4} e_ki_3 \int\limits_{\Sigma} U_k(x, y, z)dxdy + i\sum_{k=1}^{4} e_k \int\limits_{\Sigma} V_k(x, y, z)dydz+$$

$$+i\sum_{k=1}^{4} e_ki_2 \int\limits_{\Sigma} V_k(x, y, z)dzdx + i\sum_{k=1}^{4} e_ki_3 \int\limits_{\Sigma} V_k(x, y, z)dxdy.$$

If a domain $\Omega \subset \mathbb{R}^3$ has a closed piece-smooth boundary $\partial\Omega$ and a mapping $\Psi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is continuous together with partial derivatives of the first order up to the boundary $\partial\Omega_\zeta$, then the following analogues of the Gauss – Ostrogradsky formula are true:

$$\int\limits_{\partial\Omega_\zeta} \sigma\Psi(\zeta) = \int\limits_{\Omega_\zeta} \left(\frac{\partial\Psi}{\partial x} + i_2 \frac{\partial\Psi}{\partial y} + i_3 \frac{\partial\Psi}{\partial z} \right) dxdydz. \tag{15}$$

and

$$\int\limits_{\partial\Omega_\zeta} \Psi(\zeta)\sigma = \int\limits_{\Omega_\zeta} \left(\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial y} i_2 + \frac{\partial\Psi}{\partial z} i_3 \right) dxdydz. \tag{16}$$

Now, the next theorem is a result of the formulas (15), (16) and the conditions (2), (3), respectively.

Theorem 4. *Suppose that Ω has a closed piece-smooth boundary $\partial\Omega$. Suppose also that the mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a right-G-monogenic in Ω_ζ and $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ is a left-G-monogenic in Ω_ζ, and these mappings are continuous together with partial derivatives of the first order up to the boundary $\partial\Omega_\zeta$. Then*

$$\int\limits_{\partial\Omega_\zeta} \sigma\Phi(\zeta) = \int\limits_{\Omega_\zeta} (1 + i_2^2 + i_3^2)\Phi'(\zeta) dxdydz \tag{17}$$

and

$$\int\limits_{\partial\Omega_\zeta} \widehat{\Phi}(\zeta)\sigma = \int\limits_{\Omega_\zeta} \widehat{\Phi}'(\zeta)(1 + i_2^2 + i_3^2) dxdydz. \tag{18}$$

Corollary. *Under the conditions of Theorem 4 with the additional assumption $1 + i_2^2 + i_3^2 = 0$, i. e. the mappings Φ and $\widehat{\Phi}$ are solutions of the three-dimensional Laplace equation, then the equalities (17) and (18) can be rewritten in the form*

$$\int\limits_{\partial\Omega_\zeta} \sigma\Phi(\zeta) = \int\limits_{\partial\Omega_\zeta} \widehat{\Phi}(\zeta)\sigma = 0.$$

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