DOI: 10.1515/auom-2016-0038
An. Şt. Univ. Ovidius Constanţa **Vol. 24(2),2016, 217–220**

Flat local morphisms of rings with prescribed depth and dimension

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Abstract

For any pairs of integers (n, m) and (d, e) such that $0 \leq n \leq m,\ 0 \leq d \leq e,\ d \leq n,\ e \leq m$ and $n - d \leq m - e$ we construct a local flat ring morphism of noetherian local rings $u : A \to B$ such that $\dim(A) = n, \operatorname{depth}(A) = d, \dim(B) = m$ and $\operatorname{depth}(B) = e$.

1 Introduction

While preparing [3], the present author was looking for an example of a flat local ring homomorphism of noetherian local rings $u : (A, m) \to (B, n)$ such that A and B/mB are almost Cohen-Macaulay, while B is not almost Cohen-Macaulay. This means that, for example, one should construct such a morphism with $\operatorname{depth}(B) = \operatorname{depth}(A) = 0, \dim(B) = 2$ and $\dim(A) = 1$. Note that actually the flatness of the homomorphism u is the non-trivial point in the construction. After asking several people without obtaining a satisfactory answer, he decided to let it as an open question in [3]. The answer came soon, an example with the desired features being constructed by Tabaâ [6]. Using his idea, we construct a quite general example of this type, construction that can be useful in various situations.

Key Words: Flat morphism, depth, dimension.
2010 Mathematics Subject Classification: Primary 13B40; Secondary 13C15.
Received: 09.03.2015
Accepted: 20.04.2015

2 The construction

We start by pointing out the following easy and well-known fact.

Lemma 2.1. *Let k be a field, $n, d \in \mathbb{N}$ such that $0 \leq d \leq n$. Then there exists $m \in \mathbb{N}, m \geq n$ and a monomial ideal $I \subset S := k[X_1, \ldots, X_m]$ such that* $\dim(S/I)_{(X_1,\ldots,X_m)} = n$ *and* $\operatorname{depth}(S/I)_{(X_1,\ldots,X_m)} = d$.

Proof: Let $r = n-d$. If $r = 0$ the assertion is clear. Assume that $r > 0$. If $d = 0$ let $S = k[X_0, X_1, \ldots, X_r]$ and if $d > 0$ let $S = k[X_0, X_1, \ldots, X_r, T_1, \ldots, T_d]$. Consider for example the monomial ideal $I = (X_0) \cap (X_0, \ldots, X_r)^{r+1}$. Then we have $\operatorname{Ass}(S/I) = \{(X_0), (X_0, \ldots, X_r)\}$, hence $\dim(S/I) = r + d = n$ and $\operatorname{depth}(S/I) = d$. Now taking $m = r+1+d$ and renumbering the indeterminates we get the desired relations.

Remark 2.2. *Clearly there are also many other choices for a monomial ideal with the properties of the above lemma. For more about this kind of construction one can see [5].*

Theorem 2.3. *Let $0 \leq d_1 \leq n_1$ and $0 \leq d_2 \leq n_2$ be natural numbers such that $n_1 \leq n_2, d_1 \leq d_2$ and $n_1 - d_1 \leq n_2 - d_2$. Then there exists a local flat morphism of noetherian local rings $u : (A, m) \to (B, n)$ such that* $\operatorname{depth}(A) = d_1, \dim(A) = n_1, \operatorname{depth}(B) = d_2$ *and* $\dim(B) = n_2$.

Proof: Let k be a field and $A = (k[X]/I)_{(X)}, X = (X_1, \ldots, X_m)$ be a local ring obtained cf. 2.1 with $\operatorname{depth}(A) = d_1$ and $\dim(A) = n_1$. Let $s = d_2 - d_1$ and $t = n_2-n_1$. By assumption we have $s \leq t$. Hence let $C = (k[Y]/J)_{(Y)}, Y = (Y_1, \ldots, Y_p)$ be a local ring obtained cf. 2.1 with $\operatorname{depth}(C) = s$ and $\dim(C) = t$. Now let

$$D := A \otimes_k C = k[X]/I \otimes_k k[Y]/J = k[X, Y](Ik[X, Y] + Jk[X, Y])$$

and let $B := D_{(X,Y)}$. Then obviously the canonical morphism $u : A \to B$ is flat and local, being a localisation of the base change of the flat morphism $k \to C$. We need the following probably well-known fact:

Lemma 2.4. *Let k be a field and $m, p \in \mathbb{N}$. Let also I and J be monomial ideals in $k[X] = k[X_1, \ldots, X_m]$ and $k[Y] = k[Y_1, \ldots, Y_p]$ respectively and set $S := k[X, Y]$. Then* $\operatorname{Min}(IS + JS) = \{PS + QS \mid P \in \operatorname{Min}(I), Q \in \operatorname{Min}(J)\}$. *Consequently*

$$\dim(S/(IS + JS)) = \dim(k[X]/I) + \dim(k[Y]/J).$$

Proof: Using [2], 3.4 we obtain

$$\operatorname{Min}(IS + JS) = \operatorname{Min}(\sqrt{IS + JS}) = \operatorname{Min}(\sqrt{\sqrt{IS} + \sqrt{JS}}) =$$

$$= \mathrm{Min}(\sqrt{I}S + \sqrt{J}S) = \{PS + QS \mid P \in \mathrm{Min}(I), Q \in \mathrm{Min}(J)\}.$$

Returning at the proof of the Theorem, by 2.4 we get $\dim(B) = \dim(A) + \dim(C) = n_1 + t = n_2$ and by [1], Lemma 2 we have that $\mathrm{depth}(B) = \mathrm{depth}(A) + \mathrm{depth}(C) = s + d_1 = d_2$. This concludes the proof of 2.3.

Example 2.5. *Let k be a field, let $A = C = (k[X,Y]/(X^2, XY))_{(X,Y)}$ and let $B = A \otimes_k C = (k[X,Y,U,V]/(X^2, XY, U^2, UV))_{(X,Y,U,V)}$. The canonical morphism $u : A \to B$ is the morphism obtained performing the above construction. This is the example from [6], namely we have* $\dim(A) = 1, \dim(B) = 2, \mathrm{depth}(A) = \mathrm{depth}(B) = 0$.

Remark 2.6. *Let (A, m) be a noetherian local ring. Then the natural number* $\mathrm{cmd}(A) = \dim(A) - \mathrm{depth}(A)$ *is called the Cohen-Macaulay defect of A. Thus A is Cohen-Macaulay if and only if* $\mathrm{cmd}(A) = 0$ *and A is almost Cohen-Macaulay if and only if* $\mathrm{cmd}(A) \leq 1$*(see [3]).*

Example 2.7. *Using the above construction, one can also get examples of flat local morphisms of noetherian local rings, whose closed fiber has prescribed Cohen-Macaulay defect, or even more general, has prescribed dimension n and depth $d \leq n$. Indeed, by ([4], 15.1, 23.3), the flatness of u implies that $n = \dim(B/mB) = n_2 - n_1$ and $d = \mathrm{depth}(B/mB) = d_2 - d_1$, so that it is enough to choose appropriate values for $n_1 \leq n_2$ and $d_1 \leq d_2$ and perform the above construction.*

Acknowledgment: The author would like to thank Marius Vlădoiu for some illuminating discussions concerning lemma 2.4 and Javier Majadas for pointing out a missing condition in the statement of the main result.

References

[1] R. Fröberg, *A note on the Stanley-Reisner ring of a join and of a suspension*, Manuscripta Math., 60(1988), 89-91.

[2] H. T. Hà, S. Morey, *Embedded associated primes of powers of square-free monomial ideals*, J. Pure Appl. Algebra, 214(2010), 301-308.

[3] C. Ionescu, *More properties of almost Cohen-Macaulay rings*, J. Commut. Algebra, 7(2015), 363-372.

[4] H. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, 1986.

[5] R. Sharp, *Some results on the vanishing of local cohomology modules*, Proc. London Math. Soc., 30(1975), 177-195.

[6] M. Tabaâ, *Sur le produit tensoriel d'algèbres*, preprint, arxiv 1304.5395v3, arxiv.org.

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