DOI: 10.1515/auom-2016-0033
An. Şt. Univ. Ovidius Constanţa

VERSITA
Vol. 24(2),2016, 149–158

On the binomial edge ideals of block graphs

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Abstract

We find a class of block graphs whose binomial edge ideals have minimal regularity. As a consequence, we characterize the trees whose binomial edge ideals have minimal regularity. Also, we show that the binomial edge ideal of a block graph has the same depth as its initial ideal.

1 Introduction

In this paper we study homological properties of some classes of binomial edge ideals.

Let G be a simple graph on the vertex set $[n]$ and let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables over a field K. For $1 \leq i < j \leq n$, we set $f_{ij} = x_i y_j - x_j y_i$. The binomial edge ideal of G is defined as $J_G = (f_{ij} : \{i, j\} \in E(G))$. Binomial edge ideals were introduced in [8] and [12]. Algebraic and homological properties of binomial edge ideals have been studied in several papers. In [5], it was conjectured that J_G and $\mathrm{in}_<(J_G)$ have the same extremal Betti numbers. Here $<$ denotes the lexicographic order in S induced by $x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. This conjecture was proved in [3] for cycles and complete bipartite graphs. In [6], it was shown that, for a closed graph G, J_G and $\mathrm{in}_<(J_G)$ have the same regularity which can be expressed in the combinatorial data of the graph. We recall that a graph G is closed if and only if it has a quadratic Gröbner basis with respect to the lexicographic order.

Key Words: Binomial edge ideals, regularity, depth.
2010 Mathematics Subject Classification: 13D02, 05E40.
Received: 04.02.2015
Accepted: 02.03.2015

In support of the conjecture given in [5], we show, in Section 3, that if G is a block graph, then $\operatorname{depth}(S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G))$; see Theorem 3.2. By a block graph we mean a chordal graph G with the property that any two maximal cliques of G intersect in at most one vertex.

Also, in the same section, we show a similar equality for regularity. More precisely, in Theorem 3.4 we show that $\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}_<(J_G)) = \ell$ if G a C_ℓ-graph. C_ℓ-graphs constitute a subclass of the block graphs; see Section 3 for definition and Figure 1 for an example.

In [10] it was shown that, for any connected graph G on the vertex set $[n]$, we have

$$\ell \leq \operatorname{reg}(S/J_G) \leq n - 1,$$

where ℓ is the length of the longest induced path of G.

The main motivation of our work was to answer the following question. May we characterize the connected graphs G whose longest induced path has length ℓ and $\operatorname{reg}(S/J_G) = \ell$? We succeeded to answer this question for trees. In Theorem 4.1, we show that if T is a tree whose longest induced path has length ℓ, then $\operatorname{reg}(S/J_T) = \ell$ if and only if T is caterpillar. A caterpillar tree is a tree T with the property that it contains a path P such that any vertex of T is either a vertex of P or it is adjacent to a vertex of P.

In [11], the so-called weakly closed graphs were introduced. This is a class of graphs which includes closed graphs. In the same paper, it was shown that a tree is caterpillar if and only if it is a weakly closed graph. Having in mind our Theorem 4.1 and Theorem 3.2 in [6] which states that $\operatorname{reg}(S/J_G) = \ell$ if G is a connected closed graph whose longest induced path has length ℓ, and by some computer experiments, we are tempted to formulate the following.

Conjecture 1.1. *If G is a connected weakly closed graph whose longest induced path has length ℓ, then* $\operatorname{reg}(S/J_G) = \ell$.

2 Preliminaries

In this section we introduce the notation used in this paper and summarize a few results on binomial edge ideals.

Let G be a simple graph on the vertex set $[n] = \{1, \ldots, n\}$, that is, G has no loops and no multiple edges. Furthermore, let K be a field and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables. For $1 \leq i < j \leq n$, we set $f_{ij} = x_i y_j - x_j y_i$. The *binomial edge ideal* $J_G \subset S$ associated with G is generated by all the quadratic binomials $f_{ij} = x_i y_j - x_j y_i$ such that $\{i, j\} \in E(G)$. Binomial edge ideals were introduced in the papers [8] and [12].

We first recall some basic definitions from graph theory. A vertex i of G whose deletion from the graph gives a graph with more connected components than G is called a *cut point* of G. A *chordal* graph is a graph without cycles of length greater than or equal to 4. A *clique* of a graph G is a complete subgraph of G. The cliques of a graph G form a simplicial complex, $\Delta(G)$, which is called the *clique complex* of G. Its facets are the maximal cliques of G. A graph G is a *block graph* if and only if it is chordal and every two maximal cliques have at most one vertex in common. This class was considered in [5, Theorem 1.1].

The clique complex $\Delta(G)$ of a chordal graph G has the property that there exists a *leaf order* on its facets. This means that the facets of $\Delta(G)$ may be ordered as $F_1, \ldots, F_r$ such that, for every $i > 1$, F_i is a leaf of the simplicial complex generated by $F_1, \ldots, F_i$. A *leaf* F of a simplicial complex Δ is a facet of Δ with the property that there exists another facet of Δ, say G, such that, for every facet $H \neq F$ of Δ, $H \cap F \subseteq G \cap F$.

Let $<$ be the lexicographic order on S induced by the natural order of the variables. As it was shown in [8, Theorem 2.1], the Gröbner basis of J_G with respect to this order may be given in terms of the admissible paths of G. We recall the definition of admissible paths from [8].

Definition 2.1. *[8]* Let $i < j$ be two vertices of G. A path $i = i_0, i_1, \ldots, i_{r-1}, i_r = j$ from i to j is called *admissible* if the following conditions are fulfilled:

1. $i_k \neq i_l$ for $k \neq l$;
2. for each $k = 1, \ldots, r-1$ on has either $i_k < i$ or $i_k > j$;
3. for any proper subset $\{j_1, \ldots, j_s\}$ of $\{i_1, \ldots, i_{r-1}\}$, the sequence $i, j_1, \ldots, j_s, j$ is not a path in G.

Given an admissible path π in G from i to j, we set $u_\pi = (\prod_{i_k > j} x_{i_k})(\prod_{i_l < i} y_{i_l})$.

By [8, Theorem 2.1], it follows that

$$\mathrm{in}_<(J_G) = (u_\pi x_i y_j : i < j, \pi \text{ is an admissible path from } i \text{ to } j).$$

In particular, $\mathrm{in}_<(J_G)$ is a radical monomial ideal which implies that the binomial edge ideal J_G is radical as well. Hence J_G is equal to the intersection of all its minimal prime ideals. The minimal prime ideals were determined in [8, Section 3] in terms of the combinatorial data of the graph.

3 Initial ideals of binomial edge ideals of block graphs

In this section, we first show that, for a block graph G on $[n]$ with c connected components, we have $\mathrm{depth}(S/J_G) = \mathrm{depth}(S/\mathrm{in}_<(J_G)) = n + c$, where $<$

denotes the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ in the ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

We begin with the following lemma.

Lemma 3.1. *Let G be a graph on the vertex set $[n]$ and let $i \in [n]$. Then*

$$\operatorname{in}_<(J_G, x_i, y_i) = (\operatorname{in}_<(J_G), x_i, y_i).$$

Proof. We have $\operatorname{in}_<(J_G, x_i, y_i) = \operatorname{in}_<(J_{G\setminus\{i\}}, x_i, y_i) = (\operatorname{in}_<(J_{G\setminus\{i\}}), x_i, y_i)$. Therefore, we have to show that $(\operatorname{in}_<(J_G), x_i, y_i) = (\operatorname{in}_<(J_{G\setminus\{i\}}), x_i, y_i)$. The inclusion $\supseteq$ is obvious since $J_{G\setminus\{i\}} \subset J_G$. For the other inclusion, let us take u to be a minimal generator of $\operatorname{in}_<(J_G)$. If $x_i \mid u$ or $y_i \mid u$, obviously $u \in (\operatorname{in}(J_{G\setminus\{i\}}), x_i, y_i)$. Let now $x_i \nmid u$ and $y_i \nmid u$. This means that $u = u_\pi x_k y_l$ for some admissible path π from k to l which does not contain the vertex i. Then it follows that π is a path from k to l in $G\setminus\{i\}$, hence $u \in \operatorname{in}_<(J_{G\setminus\{i\}})$. $\square$

Theorem 3.2. *Let G be a block graph. Then*

$$\operatorname{depth}(S/J_G) = \operatorname{depth}(S/\operatorname{in}_<(J_G)) = n + c,$$

where c is the number of connected component of G.

Proof. Let $G_1, \ldots, G_c$ be the connected components of G and $S_i = K[\{x_j, y_j\}_{j\in G_i}]$. Then $S/J_G \cong S_1/J_{G_1} \otimes \cdots \otimes S_c/J_{G_c}$, so that

$$\operatorname{depth} S/J_G = \operatorname{depth} S_1/J_{G_1} + \cdots + \operatorname{depth} S_c/J_{G_c}.$$

Moreover, we have $S/\operatorname{in}_<(J_G) \cong S/\operatorname{in}_<(J_{G_1}) \otimes \cdots \otimes S/\operatorname{in}_<(J_{G_c})$, thus

$$\operatorname{depth} S/\operatorname{in}_<(J_G) = \operatorname{depth} S_1/\operatorname{in}_<(J_{G_1}) + \cdots + \operatorname{depth} S_c/\operatorname{in}_<(J_{G_c}).$$

Hence, without loss of generality, we may assume that G is connected. By [5, Theorem 1.1] we know that $\operatorname{depth}(S/J_G) = n + 1$. In order to show that $\operatorname{depth}(S/\operatorname{in}_<(J_G)) = n+1$, we proceed by induction on the number of maximal cliques of G. Let $\Delta(G)$ be the clique complex of G and let $F_1, \ldots, F_r$ be a leaf order on the facets of $\Delta(G)$. If $r = 1$, then G is a simplex and the statement is well known. Let $r > 1$; since F_r is a leaf, there exists a unique vertex, say $i \in F_r$, such that $F_r \cap F_j = \{i\}$ where F_j is a branch of F_r. Let $F_{t_1}, \ldots, F_{t_q}$ be the facets of $\Delta(G)$ which intersect the leaf F_r in the vertex $\{i\}$. Following the proof of [5, Theorem 1.1] we may write $J_G = J_1 \cap J_2$ where $J_1 = \bigcap_{i\notin \mathcal{S}} P_{\mathcal{S}}(G)$ and $J_2 = \bigcap_{i\in\mathcal{S}} P_{\mathcal{S}}(G)$. Then, as it was shown in the proof of [5, Theorem 1.1], it follows that $J_1 = J_{G'}$ where G' is obtained from G by replacing the cliques $F_{t_1}, \ldots, F_{t_q}$ and F_r by the clique on the vertex set $F_r \cup (\bigcup_{j=1}^{q} F_{t_j})$. Also, $J_2 = (x_i, y_i) + J_{G''}$ where G'' is the restriction of G to the vertex set $[n] \setminus \{i\}$.

We have $\mathrm{in}_<(J_G) = \mathrm{in}_<(J_1 \cap J_2)$. By [1, Lemma 1.3], we have $\mathrm{in}_<(J_1 \cap J_2) = \mathrm{in}_<(J_1) \cap \mathrm{in}_<(J_2)$ if and only if $\mathrm{in}_<(J_1 + J_2) = \mathrm{in}_<(J_1) + \mathrm{in}_<(J_2)$. But $\mathrm{in}_<(J_1+J_2) = \mathrm{in}_<(J_{G'}+(x_i, y_i)+J_{G''}) = \mathrm{in}_<(J_{G'}+(x_i, y_i))$. Hence, by Lemma 3.1, we get $\mathrm{in}_<(J_1 + J_2) = \mathrm{in}_<(J_{G'}) + (x_i, y_i) = \mathrm{in}_<(J_1) + \mathrm{in}_<(J_2)$. Therefore, we get $\mathrm{in}_<(J_G) = \mathrm{in}_<(J_1) \cap \mathrm{in}_<(J_2)$ and, consequently, we have the following exact sequence of S-modules

$$0 \longrightarrow \frac{S}{\mathrm{in}_<(J_G)} \longrightarrow \frac{S}{\mathrm{in}_<(J_1)} \oplus \frac{S}{\mathrm{in}_<(J_2)} \longrightarrow \frac{S}{\mathrm{in}_<(J_1 + J_2)} \longrightarrow 0.$$

By using again Lemma 3.1, we have $\mathrm{in}_<(J_2) = \mathrm{in}_<((x_i, y_i), J_{G''}) = (x_i, y_i) + \mathrm{in}_<(J_{G''})$. Thus, we have actually the following exact sequence

$$0 \longrightarrow \frac{S}{\mathrm{in}_<(J_G)} \longrightarrow \frac{S}{\mathrm{in}_<(J_{G'})} \oplus \frac{S}{(x_i, y_i) + \mathrm{in}_<(J_{G''})} \longrightarrow \frac{S}{(x_i, y_i) + \mathrm{in}_<(J_{G'})} \longrightarrow 0. \tag{1}$$

Since G' inherits the properties of G and has a smaller number of maximal cliques than G, it follows, by the inductive hypothesis, that

$$\mathit{depth}(S/J_{G'}) = \mathrm{depth}(S/\mathrm{in}_<(J_{G'})) = n + 1.$$

Let S_i be the polynomial ring $S/(x_i, y_i)$. Then $S/((x_i, y_i) + \mathrm{in}_<(J_{G''})) \cong S_i/\mathrm{in}_<(J_{G''})$. Since G'' is a graph on $n-1$ vertices with $q+1$ connected components and satisfies our conditions, the inductive hypothesis implies that $\mathrm{depth}\, S/((x_i, y_i) + \mathrm{in}_<(J_{G''})) = n + q \geq n + 1$. Hence,

$$\mathrm{depth}(S/\mathrm{in}_<(J_{G'}) \oplus S/((x_i, y_i) + \mathrm{in}_<(J_{G''}))) = n + 1.$$

Next, we observe that $S/((x_i, y_i) + \mathrm{in}_<(J_{G'})) \cong S_i/\mathrm{in}_<(J_H)$, where H is obtained from G' by replacing the clique on the vertex set $F_r \cup (\bigcup_{j=1}^q F_{t_j})$ by the clique on the vertex set $F_r \cup (\bigcup_{j=1}^q F_{t_j}) \setminus \{i\}$. Hence, by the inductive hypothesis, $\mathrm{depth}(S/((x_i, y_i) + \mathrm{in}_<(J_{G'}))) = n$ since H is connected and its vertex set has cardinality $n-1$. Hence, by applying the Depth lemma to exact sequence (1), we get

$$\mathrm{depth}\, S/J_G = \mathrm{depth}\, S/\mathrm{in}_<(J_G) = n + 1.$$

$\square$

Definition 3.3. Let $\ell \geq 2$ be an integer. A C_ℓ–graph is a connected graph G on the vertex set $[n]$ which consists of

(i) a sequence of maximal cliques $F_1, \ldots, F_\ell$ with $\dim F_i \geq 1$ for all i such that $|F_i \cap F_{i+1}| = 1$ for $1 \leq i \leq \ell - 1$ and $F_i \cap F_j = \emptyset$ for any $i < j$ such that $j \neq i + 1$, together with

(ii) some additional edges of the form $F = \{j, k\}$ where j is an intersection point of two consecutive cliques F_i, F_{i+1} for some $1 \leq i \leq \ell - 1$, and k is a vertex of degree 1.

In other words, G is obtained from a graph H with $\Delta(H) = \langle F_1, \ldots, F_l \rangle$ whose binomial edge ideal is Cohen-Macaulay (see [5, Theorem 3.1]) by attaching edges in the intersection points of the facets of $\Delta(H)$. Obviously, such a graph has the property that its longest induced path has length equal to ℓ. In the case that $\dim F_i = 1$ for $1 \leq i \leq \ell$, then G is called a *caterpillar graph*. Figure 1 displayes a C_ℓ–graph with $\ell = 5$..

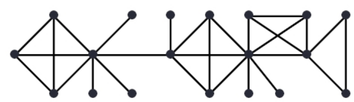

Figure 1: C_ℓ-graph

We should also note that any $\mathcal{C}_\ell$–graph is chordal and has the property that any two distinct maximal cliques intersect in at most one vertex. So that any C_ℓ-graph is a connected block graph. But, obviously, there are block graphs which are not C_ℓ–graphs. Such an example is displayed in Figure 2.

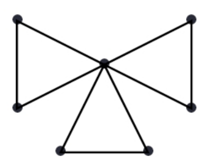

Figure 2: A block graph which is not a C_ℓ-graph

Theorem 3.4. *Let G be a $\mathcal{C}_\ell$-graph on the vertex set $[n]$. Then*

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/\operatorname{in}_<(J_G)) = \ell.$$

Proof. Let G consists of the sequence of maximal cliques $F_1, \ldots, F_\ell$ as in condition (i) in Definition 3.3 to which we add some edges as in condition (ii). So the maximal cliques of G are $F_1, \ldots, F_\ell$ and all the additional whiskers. We proceed by induction on the number r of maximal cliques of G. If $r = \ell$, then G is a closed graph whose binomial edge ideal is Cohen-Macaulay, hence the statement holds by [6, Theorem 3.2]. Let $r > \ell$ and let $F'_1, \ldots, F'_r$ be a leaf order on the facets of $\Delta(G)$. Obviously, we may choose a leaf order on $\Delta(G)$

such that $F_r' = F_\ell$. With the same arguments and notation as in the proof of Theorem 3.2, we get the sequence (1).

We now observe that G' is a $\mathcal{C}_{\ell-1}$-graph, hence, by the inductive hypothesis,

$$\operatorname{reg}\frac{S}{J_{G'}} = \operatorname{reg}\frac{S}{\operatorname{in}_<(J_{G'})} = \ell - 1. \tag{2}$$

The graph G'' has at most two non-trivial connected components. One of them, say H_1, is a $\mathcal{C}_{\ell'}$-graph with $\ell' \in \{\ell-2, \ell-1\}$. The other possible non-trivial component, say H_2, occurs if $|F_\ell| \geq 3$ and, in this case, H_2 is a clique of dimension $|F_\ell| - 2 \geq 1$. By the inductive hypothesis, we obtain

$$\operatorname{reg}\frac{S}{J_{G''}} = \operatorname{reg}\frac{S}{\operatorname{in}_<(J_{G''})} = \operatorname{reg}\frac{S}{J_{H_1}} + \operatorname{reg}\frac{S}{J_{H_2}} \leq \ell - 1 + 1 = \ell. \tag{3}$$

Relations (2) and (3) yield $\operatorname{reg}(S/\operatorname{in}_<(J_{G'}) \oplus S/((x_i, y_i) + \operatorname{in}_<(J_{G''}))) \leq \ell$. From the exact sequence (1) we get

$$\operatorname{reg}\left(\frac{S}{\operatorname{in}_<(J_G)}\right) \leq \max\{\operatorname{reg}\left(\frac{S}{\operatorname{in}_<(J_{G'})} \oplus \frac{S}{(x_i, y_i) + \operatorname{in}_<(J_{G''})}\right), \operatorname{reg}\frac{S}{\operatorname{in}_<(J_{G'})} + 1\} \leq \ell. \tag{4}$$

By [7, Theorem 3.3.4], we know that $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/\operatorname{in}_<(J_G))$, and by [10, Theorem 1.1], we have $\operatorname{reg}(S/J_G) \geq \ell$. By using all these inequalities, we get the desired conclusion. □

4 Binomial edge ideals of caterpillar trees

Matsuda and Murai showed in [10] that, for any connected graph G on the vertex set $[n]$, we have $\ell \leq \operatorname{reg}(S/J_G) \leq n-1$, where ℓ denotes the length of the longest induced path of G, and conjectured that $\operatorname{reg}(S/J_G) = n-1$ if and only if T is a line graph. Several recent papers are concerned with this conjecture; see, for example, [6], [13], and [14]. One may ask as well to characterize connected graphs G whose longest induced path has length ℓ and $\operatorname{reg}(S/J_G) = \ell$. In this section, we answer this question for trees.

A caterpillar tree is a tree T with the property that it contains a path P such that any vertex of T is either a vertex of P or it is adjacent to a vertex of P. Clearly, any caterpillar tree is a $\mathcal{C}_\ell$-graph for some positive integer ℓ.

Caterpillar trees were first studied by Harary and Schwenk [9]. These graphs have applications in chemistry and physics [4]. In Figure 3, an example of caterpillar tree is displayed. Note that any caterpillar tree is a narrow graph in the sense of Cox and Erskine [2]. Conversely, one may easily see that any narrow tree is a caterpillar tree. Moreover, as it was observed in [11], a tree is

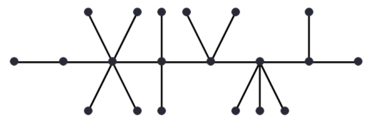

Figure 3: Caterpillar

a caterpillar graph if and only if it is weakly closed in the sense of definition given in [11].

In the next theorem we characterize the trees T with $\operatorname{reg}(S/J_T) = \ell$ where ℓ is the length of the longest induced path of T.

Theorem 4.1. *Let T be a tree on the vertex set $[n]$ whose longest induced path P has length ℓ. Then $\operatorname{reg}(S/J_T) = \ell$ if and only if T is caterpillar.*

Proof. Let T be a caterpillar tree whose longest induced path has length ℓ. Then, by the definition of a caterpillar tree, it follows that T is a $\mathcal{C}_\ell$-graph. Hence, $\operatorname{reg}(S/J_T) = \ell$ by Theorem 3.4. Conversely, let $\operatorname{reg}(S/J_T) = \ell$ and assume that T is not caterpillar. Then T contains an induced subgraph H with $\ell + 3$ vertices as in Figure 4.

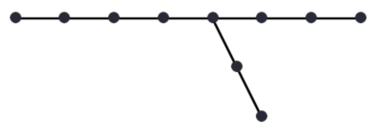

Figure 4: Induced graph H

Then, by [15, Theorem 27], it follows that $\operatorname{reg}(S/J_H) = \ell + 1$. Thus, since $\operatorname{reg}(S/J_H) \leq \operatorname{reg}(S/J_G)$ (see [10, Corollary 2.2]), it follows that $\operatorname{reg}(S/J_G) \geq \ell + 1$, contradiction to our hypothesis. $\square$

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